[10] If $f(x_1, \ldots, x_n)$ is linear in the ith place, then by $f^*_{(i)}(x_1, \ldots, x_n)$ we shall mean the adjoint of $f(x_1, \ldots, x_n)$ considered as a linear function of x_i.

[11] See the following two papers and the numerous references given there: Michal, A. D., and Hyers, D. H., "Second Order Differential Equations with Two Point Boundary Conditions in General Analysis," *Amer. Jour. Math.*, **58**, 646–660 (1936); Michal, A. D., and Elconin, V., "Completely Integrable Differential Equations in Abstract Spaces," *Acta Mathematica*, **68**, 71–108 (1937).

[12] For the notion of "function of class $C^{(n)}$" see Hildebrandt, T. H., and Graves, L. M., "Implicit Functions and Their Differentials in General Analysis," *Trans. Amer. Math. Soc.*, **129**, 127–153 (1927). By $E((x_0)_a)$ we mean the set of points $x \epsilon E$ such that $\| x - x_0 \| < a$.

[13] MH, section 2.

[14] MH, sections 5–7.

ERRATA

In "Continuous Rings and Their Arithmetics," by J. v. Neumann, *Proc. Nat. Acad. Sci.*, June, 1937, **23**, 341–349:

Due to an oversight, a line was omitted on p. 347, destroying the sense of paragraph 8. It should be corrected as follows: Insert between lines 12 and 11 from the bottom of page 347: "*integer* if $p(a) = 0$ for a suitable integer polynomial $p(x)$. An $a \epsilon \mathfrak{R}$ is a *general*"

$$X = mx + a, \quad Y = m^2y + b;$$

it is induced by the conformal group of the plane $w = f(z)$, and shows the effect on differential elements of third order. The extremals are straight lines (wide-open phenomenon), and transversality is defined by taking half the slope. The angle α of a "right angle" is therefore $\alpha = \frac{1}{2}$. The reciprocal $\alpha = 2$ defines anti-perpendicularity.

With relation to the above integral, angle (dihorn) is properly defined as

$$A = \frac{1}{2} \log \alpha;$$

so that instead of (8) we have, in any trihorn,

$$A_{12} + A_{23} + A_{31} = 0_1 A_{ji} = -A_{ij};$$

but we have preferred to use α so as to make all our formulas algebraic. For related material on conformal geometry see our earlier papers: *Proc. Intern. Congr. Math.*, **2**, 81 (1912); *Proc. Nat. Acad. Sci.* (with G. Comenetz), **22**, 303 (1936); *Science*, **85**, 480 (1937).

I wish to thank J. de Cicco and S. Gorn for assistance in writing the present paper.

For horn angles of all orders of contact we obtain a non-archimedean geometry. For second order contact the fundamental fifth order invariant includes Mullin's inversive invariant as a special case.

CONTINUOUS RINGS AND THEIR ARITHMETICS

BY J. v. NEUMANN

INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY

Communicated May 20, 1937

Introduction. 1. This note continues the analysis of the geometrical systems called *continuous geometries*, discussed in four previous notes of the author.[1] Again only results and outlines of proofs will be given, the details being reserved for subsequent publications.[2]

The main result obtained in A. T. was this: Every continuous geometry L is (lattice) isomorphic to the principal right-ideal lattice $R_{\mathfrak{R}}$ of a suitable regular ring $\mathfrak{R}$, which is uniquely determined by L, up to a (ring) isomorphism.[3] This result establishes a complete characterization (up to a lattice isomorphism) of L, by means of the purely algebraic entity $\mathfrak{R}$, and is satisfactorily complete under this aspect. Yet it is incomplete in so far as it fails to characterize those regular rings $\mathfrak{R}$, which arise in this manner from continuous geometries L. Indeed, the $R_{\mathfrak{R}}$ of a regular ring need not be a continuous geometry—it is in general merely a *complemented, modular lattice* (R. R., 710), and conversely, the above quoted result holds for all complemented, modular lattices L, subject only to a mild restriction.[4]

Hence the following problem arises:

(*) Characterize those regular rings $\mathfrak{R}$, for which the principal right-ideal lattice $R_{\mathfrak{R}}$ is a continuous geometry. They will be called *continuous rings*. Since the characterization of a continuous geometry is self dual (cf. C. G., 94, *Remark* 2, 94–96, 99), and since $R_{\mathfrak{R}}$ and $L_{\mathfrak{R}}$ are dual (anti-isomorphic, cf. R. R., 710), so both $R_{\mathfrak{R}}$ and $L_{\mathfrak{R}}$ are continuous geometries, if one is. Hence (*) is really right-left symmetric.

It follows from the above quoted result, that

(⁑) L isomorphic to $R_{\mathfrak{R}}$ establishes a (up to isomorphisms) one-to-one correspondence between all continuous geometries L and all continuous rings $\mathfrak{R}$.

In this note we will give a complete algebraic characterization of the continuous rings. We shall also see that those rings are the infinite limiting case of the *simple rings*. (§§2–4) An analysis of the notions of algebraicity and transcendency (with respect to the center), which permits applications to the theory of elementary divisors, follows (§§5–7). Finally the first steps toward an arithmetic of continuous rings are made (§8). These latter investigations reveal quite unexpected conditions: They permit to extend the notion of an integer from algebraic to transcendental elements (in a continuous ring), and they show that continuous rings are in many ways simpler than discrete ones (i.e., matrix rings over division algebras; cf. the end of §7).

The Rank. 2. Consider first a system L satisfying our axioms I–VI (C. G., 94–96), which therefore is either a discrete projective geometry of $n - 1$ dimensions: $L = L_n$, $n = 1,2, \ldots$, or a continuous geometry $L = L_\infty$ (E. C. G., 101). Assume that it is isomorphic to the $R_{\mathfrak{R}}$ of a regular ring $\mathfrak{R}$—which is certainly the case for $L = L_n$, $n = 4,5, \ldots, \infty$ (cf. A. T., 20, and[4]). We identify L with the (isomorphic) $R_{\mathfrak{R}}$, and the dual geometry L' with $L_{\mathfrak{R}}$ (which is dual to $R_{\mathfrak{R}}$, cf. R. R., 710). We will denote the dimension functions of L and L' by $D(\mathfrak{a})$, $\mathfrak{a}\epsilon L$, and $D'(\mathfrak{a}')$, $\mathfrak{a}'\epsilon L'$, respectively.

We can prove:

(1) For every $a\epsilon\mathfrak{R}$

$$D((a)_r) = D'((a)_l) = 1 - D((a)_l^r) = 1 - D'((a)_r^l).^5$$

We therefore define:

(2) Denote the common value of the four quantities enumerated in (1) by $R(a)$, and call it the *rank of a*.

The range of $R(a)$ is the same as the range of $D(\mathfrak{a})$, that is, the set $D_n =$

$\left(0, \frac{1}{n}, \frac{2}{n} \ldots, 1\right)$ for $L = L_n$, $n = 1, 2, \ldots,$[6] and the set D_∞ of all real numbers $\geqq 0$, $\leqq 1$ for $L = L_\infty$ (C. G. 98, E. C. G. 101).

A simple analysis discloses that the rank $R(a)$ possesses the following properties:[7]

(3) (α) Always $0 \leqq R(a) \leqq 1$.
(β) $R(a) = 0$ if and only if $a = 0$.
(γ) $R(a) = 1$ if and only if a^{-1} exists.
(δ) $R(a) = R(b)$ if and only if $a = ubv$, where u^{-1}, v^{-1} exist.
(ϵ) $R(ab) \leqq \text{Min}\,(R(a), R(b))$.
(η) $R(a + b) \leqq R(a) + R(b)$.
(θ) For $e^2 = e, f^2 = f, ef = fe = 0$ we have $R(e + f) = R(e) + R(f)$.

Also:

(4) Even the following requirements, which are weakened forms of (3), (α)–(θ), possess the only solution $\overline{R}(a) \equiv R(a)$:

($\bar{\alpha}$) Always $0 \leqq \overline{R}(a) \leqq 1$.
($\bar{\beta}$) $\overline{R}(0) = 0$.
($\bar{\gamma}$) $\overline{R}(1) = 1$.
($\bar{\epsilon}_r$) $\overline{R}(ab) \leqq \overline{R}(a)$.
($\bar{\epsilon}_l$) $\overline{R}(ab) \leqq \overline{R}(b)$.
($\bar{\theta}$) For $e^2 = e, f^2 = f, ef = fe = 0$ we have $\overline{R}(e + f) = \overline{R}(e) + \overline{R}(f)$.

It suffices to require only one of the two conditions ($\bar{\epsilon}_r$) and ($\bar{\epsilon}_l$). Since (4) refers only to the ring operations in $\mathfrak{R}$, and since it is right-left symmetric, we can infer:

(5) Every (ring) automorphism or anti-automorphism of $\mathfrak{R}$ leaves $R(a)$ invariant.

3. We define the *rank-distance* of two $a, b \epsilon \mathfrak{R}$ to be $R(a - b)$. One derives immediately from (3) that this is a *metric* in $\mathfrak{R}$:

(i) $R(a - b) \begin{cases} = 0 \text{ for } a = b \\ > 0 \text{ for } a \neq b, \end{cases}$
(ii) $R(a - b) = R(b - a)$,
(iii) $R(a - c) \leqq R(a - b) + R(b - c)$,

and that $a + b$, $a \cdot b$ fulfil Lipschitz conditions:

(iv) $R((a + b) - (c + d)) \leqq R(a - c) + R(b - d)$,
(v) $R(ab - cd) \leqq R(a - c) + R(b - d)$.

For $L = L_n$, $n = 1, 2, \ldots, R(a - b)$ does not lead to an interesting topology in $\mathfrak{R}$ because its values are $0, \frac{1}{n}, \frac{2}{n} \ldots, 1$ only—but for $L = L_\infty$, where its values are all real numbers $\geqq 0, \leqq 1$, it becomes topologically significant.

$\mathfrak{R}$ is *complete* in the topology of the rank distance, that is, we can prove this:

(6) If $a_1, a_2, \ldots \epsilon\mathfrak{R}$, then the existence of an $a\epsilon\mathfrak{R}$ with

$$\lim_{\rho \to \infty} R(a_\rho - a) = 0,$$

which will be denoted by

$$\lim_{\rho \to \infty}{}^{(R)} (a_\rho) = a,$$

is equivalent to

$$\lim_{\rho,\sigma \to \infty} R(a_\rho - a_\sigma) = 0.$$

The topology of the rank distance gives us also a means to characterize the principal right (left) ideals among all right (left) ideals of $\mathfrak{R}$ (cf. R. R., 707 [Def. 1]). We can show:

(7) A right (left) ideal of $\mathfrak{R}$ is a principal one (and hence an element of $R_{\mathfrak{R}} = L$ $(L_{\mathfrak{R}} = L')$) if and only if it is a closed set in the topology of the rank distance.

The really significant case for (6), (7) is of course the continuous one: $L = L_\infty$. In the discrete cases $L = L_n$, $n = 1,2, \ldots$, this topology degenerates, and so (6) becomes vacuous, while (7) states that then all right (left) ideals are principal ones (cf. A. T., 18).

Characterization of the Continuous Rings. 4. Let us start at the opposite end. Let a regular ring $\mathfrak{R}$ be given. We assume that it is irreducible, that is, that its center $\mathfrak{Z}$ is a division algebra. (Cf. R. R., 712, the $\mathfrak{R}$ of our §§2–3 was irreducible along with L.) We define:

(8) $\mathfrak{R}$ is a *rank ring*, if a numerical function $\overline{R}(a)$ can be defined for all $a\epsilon\mathfrak{R}$, which possesses the properties (4), $(\overline{\alpha})$–$(\overline{\theta})$, with the following provisos: $(\overline{\beta})$ has to be replaced by the stronger condition

$$(\overline{\overline{\beta}})\quad \overline{R}(a) = 0 \text{ if and only if } a = 0.$$

Both conditions $(\overline{\epsilon}_r)$ and $(\overline{\epsilon}_l)$ must be required. We can show that the requirements of (8) imply also the equivalent of (3) (η):

$$(\overline{\eta})\quad \overline{R}(a + b) \leqq \overline{R}(a) + \overline{R}(b).$$

This permits us to infer that $\overline{R}(a - b)$, which we will again call a *rank distance*, is a *metric* in $\mathfrak{R}$: That is, that (i)–(iii) (at the beginning of §3) hold. Therefore we can define further:

(9) The $\mathfrak{R}$ of (8) is *complete*, if it is complete in the topology of a rank distance $\overline{R}(a - b)$, as described in (6).

We are now able to formulate our main result:

(10) For a regular (and irreducible) ring $\mathfrak{R}$ the principal right ideal lattice $R_{\mathfrak{R}}$ (and also its dual, the principal left-ideal lattice $L_{\mathfrak{R}}$) fulfils our axioms I–VI (C. G., 94–96) if and only if $\mathfrak{R}$ is a *complete rank ring.*

(11) If this is the case, then the rank $\bar{R}(a)$ of (8) is unique, and coincides with the rank $R(a)$ of (2). Hence all statements of §§2–3 hold for it. Now we see that $L = R_{\mathfrak{R}}$ is either an $L = L_n$, $n = 1,2, \ldots$, or $L = L_\infty$. In the first case $\mathfrak{R}$ is an nth order matrix algebra over a suitable division algebra $\mathfrak{d}$ (cf.[6]): $\mathfrak{R} = \mathfrak{d}_n$, and we call $\mathfrak{R}$ a *discrete ring.* In the second case $\mathfrak{R}$ is a *continuous ring*, as defined in (*) in §1.

The discrete rings $\mathfrak{R}$ coincide, by one of Wedderburn's famous theorems,[8] with all *simple rings.* Hence our notion of a continuous ring is the infinite limiting case of Wedderburn's notion of a simple ring.

Algebraic and Transcendental Elements. 5. From now on $\mathfrak{R}$ will denote a fixed complete rank ring. Let again be $L = R_{\mathfrak{R}}$, $L' = L_{\mathfrak{R}}$, $R(a)$ the (unique) rank. Later on we will even require that this $\mathfrak{R}$ be a continuous ring.

For a discrete $\mathfrak{R}$, that is $\mathfrak{R} = \mathfrak{d}_n$ at most n^2 elements of $\mathfrak{R}$ can be linearly independent with respect to $\mathfrak{d}$; hence for any $a \epsilon \mathfrak{R}$ a linear relation, with coefficients from $\mathfrak{d}$, must exist between the powers $1, a, a^2, \ldots$ of a:

(π) $p(a) = 0$, where $p(x) \equiv x^l + \alpha_1 x^{l-1} + \ldots + \alpha_l$,

(π_1) $\alpha_1, \ldots, \alpha_l \;\epsilon\; \mathfrak{d}$.

If $\mathfrak{d}$ has a finite (linear) order over its own center, which is also the center $\mathfrak{Z}$ of $\mathfrak{R}$, then the same argument applies to $\mathfrak{Z}$ instead of $\mathfrak{d}$, that is, every $a \epsilon \mathfrak{R}$ fulfils even an equation (π) with

(π_2) $\alpha_1, \ldots, \alpha_l \epsilon \mathfrak{Z}$.

But for an arbitrary $\mathfrak{d}$ this need not be true.

Observe that (π), (π_2) leads to a much more satisfactory theory than (π), (π_1): The familiar theory of polynomial equations applies in the first case only, since only $\mathfrak{Z}$ is commutative.

In a continuous $\mathfrak{R}$ this entire argument breaks down, since it contains any number of linearly independent elements. It is also discouraging that only $\mathfrak{Z}$ (the center of $\mathfrak{R}$), but no $\mathfrak{d}$, can be defined for a continuous $\mathfrak{R}$ (cf. A. T., 21–22, §7)—yet only (π), (π_1) is generally true, and it involves $\mathfrak{d}$.

6. In spite of these unfavorable symptoms, we will base our discussion in a continuous $\mathfrak{R}$ on its center $\mathfrak{Z}$. Let P be the set of all polynomials

(P) $p(x) \equiv x^l + \alpha_1 x^{l-1} + \ldots + \alpha_l,\ \alpha_1, \ldots, \alpha_l \;\epsilon\; \mathfrak{Z}$.

If $a \epsilon \mathfrak{R}$, then a $p(x) \epsilon P$ with $p(a) = 0$ need not exist. Let us therefore investigate how near we can get to $p(a) = 0$, that is, how small we can make $R(p(a))$. The most unfavorable case covers those $p(x) \epsilon P$, for which

(a) $R(p(a)) = 1$, that is $p(a)^{-1}$ exists. The other $p(x) \epsilon P$ have

(b) $R(p(a)) < 1$, that is $p(a)^{-1}$ does not exist, those we call *a-singular.* Among them the irreducible ones (with respect to the coefficient domain $\mathfrak{Z}$) are particularly important. Define:

(12) $\mathfrak{Z}$ is the set of all irreducible, a-singular polynomials from P.

Several relations between a, P, $\mathfrak{Z}$ hold both for discrete and continuous $\mathfrak{R}$ and can be established with little difficulty. We enumerate them:

(13) $\mathfrak{Z}$ is enumerable.[9] Denote its elements by $q_1(x)$, $q_2(x)$, ...

(14) Let α be the *gr.l.b.* of all $R(p(a))$, for all $p(x)\epsilon P$.[10] Let $\mathfrak{a}(\mathfrak{a}')$ be the intersection of all $(p(a))_r$, $(((p(a))_l)$, for all $p(x)\epsilon P$. Then there exists a unique *ip.* $e\epsilon\mathfrak{R}$ with

$$(e)_r = \mathfrak{a},\ (e)_l = \mathfrak{a}',$$

We have $R(e) = D(\mathfrak{a}) = D'(\mathfrak{a}') = \alpha$.

(15) Define α_i, $\mathfrak{a}_i$, $\mathfrak{a}'_i$ $(i = 1,2, \ldots)$ as the α, $\mathfrak{a}$, $\mathfrak{a}'$ in (14), but restricting $p(x)$ to the $(q_i(x))^t$, $t = 1,2, \ldots$. Then there exists a unique *ip.* $e_i\epsilon\mathfrak{R}$ with

$$(e_i)_r = \mathfrak{a}_i,\ (e_i)_l = \mathfrak{a}'_i).$$

We have $R(e_i) = D(\mathfrak{a}_i) = D'(\mathfrak{a}'_i) = \alpha_i$.

Put $\beta_i = 1 - \alpha_i$, $f_i = 1 - e_i$; so $R(f_i) = \beta_i$.

(16) We have $ef_i = f_ie = 0, f_if_j = 0$ for $i \neq j$, and $e + \Sigma f_i = 1$.[11] Also $\alpha + \sum_i \beta_i = 1$. Hence the rings $\mathfrak{R}(e)$, $\mathfrak{R}(f_i)$ $(i = 1,2, \ldots)$ (cf. R. R., 713) are mutually orthogonal.

(17) e and all f_i commute with every x which commutes with a, hence in particular with a. For every such x a unique decomposition

$$x = x_e + x_{1-e} = x_e + \sum_i x_{fi},^{11}\ x_e\epsilon\mathfrak{R}(e),\ x_{1-e}\epsilon\mathfrak{R}(1 - e),\ x_{fi}\epsilon\mathfrak{R}(f_i)$$

exists.

(18) For every $p(x)\epsilon P$ the $p(a_e)$, when formed in $\mathfrak{R}(e)$, possesses a $p(a_e)^{-1}$ in $\mathfrak{R}(e)$.

(19) Form $p(a_{1-e})$ in $\mathfrak{R}(1 - e)$; then $R(p(a_{1-e}))$ can be made arbitrarily small.

(20) Form $(q_i(a_{fi}))^t$ $(t = 1,2, \ldots)$ in $\mathfrak{R}(f_i)$, then

$$\lim_{t\to\infty} R((q_i(a_{fi}))^t) = 0.$$

A more qualitative interpretation of (13)–(20): Decompose a into $a_e + a_{1-e}$ or $a_e + \sum_i a_{fi}$. Then a_e is the *purely transcendental* part of a: In $\mathfrak{R}(e)$ every $p(a_e)$ is in the extreme case (a) (cf. above). a_{1-e} is the arbitrarily nearly algebraical part of a. Breaking it up further into the a_{fi}, we see even which irreducible polynomials express the algebraicity of a best. (13)–(20) can be used to build up a theory of proper values and of elementary divisors in $\mathfrak{R}$.

7. We obtained in §6 a complete answer to this question: How nearly 0 can $p(a)$ $(p(x)\epsilon P)$ be? That is: How small can $R(p(a))$ become? A different problem is this: How closely can a be approximated by elements a'

which are algebraic (with respect to $\mathfrak{Z}$)? That is: How small can $R(a-a')$ become for such elements a', for which $p'(x)\epsilon P$ with $p'(a') = 0$ exists?

We can answer this question, and it is remarkable that the answer is more satisfactory for continuous $\mathfrak{R}$'s than for discrete ones. We can prove by a rather involved analysis:

(21) If $\mathfrak{R}$ is continuous, then the algebraic elements (with respect to $\mathfrak{Z}$) are everywhere dense in $\mathfrak{R}$ (in the rank metric). That is: Given $a\epsilon\mathfrak{R}$ and $\epsilon > 0$, an $a'\epsilon\mathfrak{R}$ and a $p'(x)\epsilon P$ with

$$R(a - a') \leqq \epsilon, p'(a') = 0$$

exist. For a discrete $\mathfrak{R}$ (21) need not hold. In fact, if $\mathfrak{R} = \mathfrak{d}_n$, $\mathfrak{Z}$ = Center of $\mathfrak{R}$ = Center of $\mathfrak{d}$, then (21) is certainly false, if $\mathfrak{d}$ is not algebraic over its own center, $\mathfrak{Z}$![12]

Let us observe, finally, that every continuous $\mathfrak{R}$ contains "purely transcendental" a's, that is, a's with $\alpha = 1$, $\mathfrak{a} = \mathfrak{a}' = \mathfrak{R}$ (cf. (14)). For such an a $(p(a))^{-1}$ exists for all $p(x)\epsilon P$.

Integers. 8. Assume that a notion of *integers* is defined in $\mathfrak{Z}$. That is:

(I) Let a subring $\mathfrak{Y}$ of $\mathfrak{Z}$ be given, of which $\mathfrak{Z}$ is the quotient algebra. Use this terminology: The $a\epsilon\mathfrak{Z}$ are *rational*, the $a\epsilon\mathfrak{Y}$ are *rational integers*, the $a\epsilon\mathfrak{Y}$ with $a^{-1}\epsilon\mathfrak{Y}$ are *units*. An $a\epsilon\mathfrak{Y}$ which is no unit, but such that for $a = bc$, $b, c\epsilon\mathfrak{Y}$ either b or c is a unit, is a *prime*. a, b are *equivalent*, if ab^{-1} is a unit.

(II) Assume that every $a\epsilon\mathfrak{Y}$ can be written as

$$a = b_1 \ldots b_l, \quad b_1 \ldots b_l \text{ primes},$$

and that this representation is essentially (that is, up to permutations of the $b_1 \ldots b_l$, and replacement by equivalent ones) unique. We then define:

(22) A polynomial $p(x)\epsilon P$ that is

$$p(x) = x^l + \alpha_1 x^{l-1} + \ldots + \alpha_l, \alpha_1, \ldots, \alpha_l\epsilon\mathfrak{Z}$$

is *integer*, if the $\alpha_1, \ldots, \alpha_l$ are rational integers. An $a\epsilon\mathfrak{R}$ is an *algebraic integer*, if it is a (rank metric) limit point of algebraic integers. Observe how unreasonable this last definition would be, if we were dealing with real (or complex) numbers, and their usual (absolute value) metric: Every number would be a general integer in this sense. Here, however (that is in $\mathfrak{R}$ with its rank metric), we obtain a definite arithmetic structure, with properties which are obtained only after a detailed investigation.

We can prove:

(23) An $a\epsilon\mathfrak{R}$ is a general integer, if and only if all elements $q_1(x)$, $q_2(x)$, ... of $\mathfrak{J}$ in (12) are integer polynomials.

So the purely transcendental part of a (a_e in (18)) has no influence on a's arithmetical character.

From (23) we can derive rather directly:

(24) A general integer is algebraic, if and only if it is an algebraic integer.[13]

(25) A general integer is rational, if and only if it is a rational integer.[13]

We will develop the arithmetic based on these notions in subsequent publications.

[1] "Continuous Geometries," *Proc. Nat. Acad. Sci.*, **22,** 92–100 (1936); "Examples of Continuous Geometries," *Ibid.*, **22,** 101–108 (1936); "On Regular Rings," *Ibid.*, **22,** 707–713 (1936); "Algebraic Theory of Continuous Geometries," *Ibid.*, **23,** 16–22 (1937). These notes will be referred to as C. G.; E. C. G.; R. R.; A. T., respectively.

[2] Detailed accounts of the contents of the four notes quoted above,[1] with full proofs, were given by the author in lectures given in Princeton, in the years 1935–36 and 1936–37. Complete notes of these lectures were mimeographed. The entire theory will be presented as an Amer. Math. Soc. Colloquium Lecture in September, 1937, and also in book form in the Amer. Math. Soc. Colloquium Series.

[3] Cf. A. T., 20–21. "Regularity" is defined in R. R., 708, the lattice $R_{\mathfrak{R}}$ in R. R., 709–710, and A. T., 19.

[4] They must possess an "order" $\geqq 4$, cf. A. T., 20. This condition is necessary, in order to exclude "non-Desarguesian" (discrete, projective) plane geometries, which have only the order 3. Cf. loc. cit. above.

[5] For the notations cf. R. R., 707 (Def. 1) and 709 (Def. 5). As to the use of D and D', observe that $(a)_r, (a)_l^r \epsilon R_{\mathfrak{R}} = L$, and $(a)_l, (a)_r^l \epsilon L_{\mathfrak{R}} = L'$.

[6] In this case $\mathfrak{R}$ is the nth order matrix algebra over a suitable (not necessarily commutative but associative) division algebra $\mathfrak{d} : \mathfrak{R} = \mathfrak{d}_n$ (A. T., 17–18). Since $a \epsilon \mathfrak{R}$ is an nth order matrix over $\mathfrak{d}$, it possesses a matrix rank in the usual sense: $r(a) = 0,1, \ldots, n$ This is connected with the common geometrical dimension $d(\mathfrak{a}) = -1,0,1, \ldots, n-1$ of the linear subspace $\mathfrak{a}$ in L $(= L_n = R_{\mathfrak{R}})$ by the relation $r(a) = d((a)_r) + 1$. These $d(\mathfrak{a})$, $r(a)$ give our $D(\mathfrak{a})$, $R(a)$ by $D(\mathfrak{a}) = \dfrac{d(\mathfrak{a}) + 1}{n}$ (cf. C. G., footnote[13]) and $R(a) = \dfrac{r(a)}{n}$. So our rank $R(a)$ differs from the usual one $r(a)$ merely by the normalizing factor $\dfrac{1}{n}$.

[7] $R(a)$ is not an "absolute value" ("Bewertung") in the sense in which this notion is commonly used in algebra: For this the property (3), (ϵ) is too weak. For an "absolute value"

$$(\epsilon^*) \quad R(ab) \leqq R(a)\, R(b)$$

would be necessary. (3), (η), might do, although in most cases (e.g., in "p-adic" systems) even

$$(\eta^*) \quad R(a + b) \leqq Max + (R(a), R(b))$$

holds. Cf. J. Kurschak, *Jour. f. Math.*, **142,** 211–253 (1913); A. Ostrowski, *Acta Math.*, **41,** 271–284 (1918), and for the modern literature on this subject, e.g., W. Krull, *Jahresb. d. D. M. V.*, **46,** 153–171 (1936).

[8] J. H. Maclaggan Wedderburn, *Proc. London Math. Soc.*, **6,** 77–118 (1908), particularly 81, 98. See also v. d. W. II (cf. R. R., footnote[1]), 170.

[9] That is: Empty, or finite, or enumerably infinite.

[10] This number α ($\geqq 0$, $\leqq 1$) gives a quantitative estimate of how "nearly algebraic" a is, that is, how nearly $p(a) = 0$, $R(p(a)) = 0$, can be approximated.

[11] If the sum Σ is infinite at all (cf.[9]), then it converges with respect to the rank metric.

[12] In this case (21) with an $\epsilon \leqq 1/n$ requires $a = a'$, that is $p'(a) = 0$, that is the algebraicity (with respect to $\mathfrak{Z}$) of a itself.

[13] These are justifications of our use of the words "general integer." Note that these results would not hold if we considered real (or complex) numbers in their usual metric (cf. above, following (22)).