THE CONSISTENCY OF THE AXIOM OF CHOICE AND OF THE GENERALIZED CONTINUUM-HYPOTHESIS

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THEOREM. *Let* T *be the system of axioms for set-theory obtained from v. Neumann's system* S*[1] *by leaving out the axiom of choice* (Ax. III 3*); *then, if* T *is consistent, it remains so, if the following propositions* 1–4 *are adjoined simultaneously as new axioms:*

1. The axiom of choice (i.e., v. Neumann's Ax. III 3*)
2. The generalized Continuum-Hypothesis (i.e., the statement that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ holds for any ordinal α)
3. The existence of linear non-measurable sets such that both they and their complements are one-to-one projections of two-dimensional complements of analytic sets (and which therefore are B_2-sets in Lusin's terminology[2])
4. The existence of linear complements of analytic sets, which are of the power of the continuum and contain no perfect subset.

A corresponding theorem holds, if T denotes the system of Princ. Math.[3] or Fraenkel's system of axioms for set theory,[4] leaving out in both cases the axiom of choice but including the axiom of infinity.

The proof of the above theorems is constructive in the sense that, if a contradiction were obtained in the enlarged system, a contradiction in T could actually be exhibited.

The method of proof consists in constructing on the basis of the axioms of T[5] a model for which the propositions 1–4 are true. This model, roughly speaking, consists of all "mathematically constructible" sets, where the term "constructible" is to be understood in the semiintuitionistic sense which excludes impredicative procedures. This means "constructible" sets are defined to be those sets which can be obtained by Russell's ramified hierarchy of types, if extended to include transfinite orders. The extension to transfinite orders has the consequence that the model satisfies the impredicative axioms of set theory, because an axiom of reducibility can be proved for sufficiently high orders. Furthermore the proposition "Every set is constructible" (which I abbreviate by "A") can be proved to be consistent with the axioms of T, because A turns out to be true for the model consisting of the constructible sets. From A the propositions 1–4 can be deduced. In particular, proposition 2 follows from the fact that all constructible sets of integers are obtained already for orders $<\omega_1$, all constructible sets of sets of integers for orders $<\omega_2$ and so on.

The proposition A added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way. In this connection it is important that the consistency-proof for A does not break down if stronger axioms of infinity (e.g., the existence of inaccessible numbers) are adjoined to T. Hence the consistency of A seems to be absolute in some sense, although it is not possible in the present state of affairs to give a precise meaning to this phrase.

[1] Cf. *J. reine angew. Math.*, **160**, p. 227.

[2] Cf. N. Lusin, *Leçons sur les ensembles analytiques*, Paris, 1930, p. 270.

[3] Cf. A. Tarski, *Mh. Math. Phys.*, **40**, p. 97.

[4] Cf. A. Fraenkel, *Math. Zeit.*, **22**, p. 250.

[5] This means that the model is constructed by essentially transfinite methods and hence gives only a relative proof of consistency, requiring the consistency of T as a hypothesis.

THE CHARACTERIZATION OF PSEUDO-$S_{n,r}$ SETS

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I. If to each pair of elements (*points*) p, q of an abstract set there is attached a non-negative real number (*distance*) pq, independent of the order of the elements, while $pq = 0$ if and only if $p = q$, the resulting space is called *semimetric*. A fundamental problem in the distance geometry of the n-dimensional spherical surface $S_{n,r}$ of radius r(the n-dimensional surface of a sphere of radius r in euclidean space of $n + 1$ dimensions, with "*shorter arc*" distance) consists in characterizing those semimetric spaces S which have the following properties: (1) S contains more than $n + 3$ points, (2) if $p, q \epsilon S$, then $pq \neq d = \pi r$, (3) if $p_1, p_2, \ldots, p_{n+2}$ are elements of S, then there exists a function f mapping these $n + 2$ points upon $S_{n,r}$ with preservation of distances (i.e., *congruently*), (4) S cannot be mapped congruently upon a subset of $S_{n,r}$. Reserving the details of the investigation for publication elsewhere, we summarize in this note the complete solution of this problem. Semimetric spaces S with properties (3), (4) are called pseudo-$S_{n,r}$ sets.[1]

The properties of the $S_{n,r}$, by virtue of which the characterization theorems of pseudo-$S_{n,r}$ sets are obtained, are all consequences of the following *metric* ones: (1) the mutual distances of each set of $n + 2$ points of $S_{n,r}$ satisfy a relation of the form $|\varphi(p_ip_j/r)| = 0$, $(i, j = 1, 2, \ldots, n + 2)$, where $\varphi(pq/r)$ is a real, single-valued, monotonically de-