The proposition A added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way. In this connection it is important that the consistency-proof for A does not break down if stronger axioms of infinity (e.g., the existence of inaccessible numbers) are adjoined to T. Hence the consistency of A seems to be absolute in some sense, although it is not possible in the present state of affairs to give a precise meaning to this phrase.

[1] Cf. *J. reine angew. Math.*, **160,** p. 227.

[2] Cf. N. Lusin, *Leçons sur les ensembles analytiques,* Paris, 1930, p. 270.

[3] Cf. A. Tarski, *Mh. Math. Phys.*, **40,** p. 97.

[4] Cf. A. Fraenkel, *Math. Zeit.*, **22,** p. 250.

[5] This means that the model is constructed by essentially transfinite methods and hence gives only a relative proof of consistency, requiring the consistency of T as a hypothesis.

THE CHARACTERIZATION OF PSEUDO-$S_{n,r}$ SETS

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I. If to each pair of elements (*points*) p, q of an abstract set there is attached a non-negative real number (*distance*) pq, independent of the order of the elements, while $pq = 0$ if and only if $p = q$, the resulting space is called *semimetric*. A fundamental problem in the distance geometry of the n-dimensional spherical surface $S_{n,r}$ of radius r(the n-dimensional surface of a sphere of radius r in euclidean space of $n + 1$ dimensions, with "*shorter arc*" distance) consists in characterizing those semimetric spaces S which have the following properties: (1) S contains more than $n + 3$ points, (2) if $p, q \epsilon S$, then $pq \neq d = \pi r$, (3) if $p_1, p_2, \ldots, p_{n+2}$ are elements of S, then there exists a function f mapping these $n + 2$ points upon $S_{n,r}$ with preservation of distances (i.e., *congruently*), (4) S cannot be mapped congruently upon a subset of $S_{n,r}$. Reserving the details of the investigation for publication elsewhere, we summarize in this note the complete solution of this problem. Semimetric spaces S with properties (3), (4) are called pseudo-$S_{n,r}$ sets.[1]

The properties of the $S_{n,r}$, by virtue of which the characterization theorems of pseudo-$S_{n,r}$ sets are obtained, are all consequences of the following *metric* ones: (1) the mutual distances of each set of $n + 2$ points of $S_{n,r}$ satisfy a relation of the form $|\varphi(p_ip_j/r)| = 0$, $(i, j = 1, 2, \ldots, n + 2)$, where $\varphi(pq/r)$ is a real, single-valued, monotonically decreasing function defined over the *distance set* of $S_{n,r}$, with $\varphi(0) = 1$ and $\varphi(d/r) = -1$, ($\varphi(pq/r) \equiv \cos pq/r$), (2) for each integer k, and each set of $k + 1$ points of $S_{n,r}$, the above determinant is non-negative, while there exists a set of $n + 1$ points for which the determinant is positive, (3) each set of $n + 1$ points with a non-vanishing determinant forms a *complete metric basis*, (4) if $p_1, p_2, \ldots, p_n, p$ are $n + 1$ points of $S_{n,r}$, with non-vanishing determinant, there exists at least one point p', distinct from p, such that $pp_i = p'p_i$, $(i = 1, 2, \ldots, n)$.

II. *Characterization Theorems.*—The two following characterization theorems are obtained.

FIRST CHARACTERIZATION THEOREM. *If* P *is a pseudo-*$S_{n,r}$ *set containing more than* n + 3 *points and without diametral points* (*i.e., no pair of points of* P *has a distance equal to* d), *then every pair of distinct points of* P *has the distance* $r \cdot \mathrm{Cos}^{-1}(\pm 1/(n + 1))$. *Not every distance equals* $r \cdot \mathrm{Cos}^{-1}(1/(n + 1))$.

SECOND CHARACTERIZATION THEOREM. *Let* P *be a pseudo-*$S_{n,r}$ *set containing more than* n + 3 *points and without diametral points. Then for every positive integer* k, *the determinant* $|\cos p_ip_j/r|$, $(i, j = 1, 2, \ldots, k + 1)$, *formed for* k + 1 *points of* P *has* (*upon multiplication of appropriate rows and the same numbered columns by* -1) ALL *elements outside the principal diagonal equal to* $-1/(n + 1)$.

Thus, a pseudo-$S_{n,r}$ set containing more than $n + 3$ points, and without diametral points, is *essentially* equilateral, with distance $r \cdot \mathrm{Cos}^{-1}(-1/(n+1))$. It is, of course, obvious that a set of arbitrary power exceeding $n + 2$ and with every distance equal to $r \cdot \mathrm{Cos}^{-1}(-1/(n + 1))$ is a pseudo-$S_{n,r}$ set without diametral points, but that *every* pseudo-$S_{n,r}$ set *containing more than* n + 3 *points*, and without diametral points, has essentially this simple structure is surprising. Many interesting conclusions follow from this result.

[1] For $n = 1$ the term pseudo-d-cyclic is used. The characterization of pseudo-d-cyclic sets has been given (Blumenthal, L. M., *Amer. Jour. Math.*, **54**, 387–396, 729–738 (1932); **56**, 225–232 (1934) but the treatment for higher dimensions demands quite different methods. The case $n = 2$ is solved in *Distance Geometries*, University of Missouri Studies, 1938. The general case involves several departures from the procedure adopted there.