[5] [8], formula (7.17); [9], formula (2); for the case $h = 0$, [2], formula (2).

[6] The dot denotes the scalar product of vectors. The exponent two (formulas 14, 15) will denote the scalar product of a vector by itself, i.e., the sum of the squares of its components.

[7] Except that we interpret δ as a *derivative*, rather than a differential, as is more customary.

[8] Formulas (41) and (33) of [2], reproduced as (34) and (37) of the foregoing note in the present issue of these PROCEEDINGS. This is for a general topological form of the Riemann surface $\mathfrak{R}$. With increasing complexity of $\mathfrak{R}$, the appropriate identity involved successively: algebraic functions, trigonometric functions, elliptic functions, θ-functions. See, respectively [4], p. 243; [7], formula (5.4); [5], formula (7.1); [2], formulas (41), (33). In [7], the functions actually appearing are hyperbolic, due to the rotation through a right angle of the parallel strip representing the Riemann surface R.

[9] Given in [3], and in preceding abstracts in *Bull. Amer. Math. Soc.*, **36**, 50 (1930).

[10] We suppose that the contact of the tangents t, $\bar{t}$ to $\mathbf{A}$ at Q, $\bar{Q}$, respectively, is ordinary two-point contact. Higher contact can always be avoided by a preliminary birational transformation.

[11] Or its conformally equivalent orthogonal projections $\mathfrak{R} = \mathfrak{R}_x$ or $\mathfrak{R}_y$ (see art. 2). The notation is supposed arranged so that the semi-surface R contains the point Q.

[12] We may again remark that this one-to-one character breaks down in the vicinity of the points of contact Q, $\bar{Q}$ of the tangents t, $\bar{t}$. Otherwise, it extends beyond the real branches to the rest of the Riemann surfaces involved, with deletion of the stated neighborhoods.

THE MOST GENERAL FORM OF THE PROBLEM OF PLATEAU

BY JESSE DOUGLAS

BROOKLYN, NEW YORK

Communicated June 28, 1938

1. The method of the preceding note—cited hereafter as Note II—applies with practically no modification to the following "most general formulation" of the problem of Plateau, which, I believe, is given here explicitly for the first time.

PROBLEM P. *Given any riemannian manifold* R *in the most general sense of the term, i.e., any two-dimensional connected topological variety for which there is defined in the neighborhood of each point a local conformal representation on a circle.*[1] R *may then have any finite or infinite number of boundaries, and any topological structure whatever, i.e., any finite or infinite type of connectivity. It may also have either character of orientability, i.e., one- or two-sidedness.*[2]

Given also any point-set Γ *in* n*-dimensional euclidean space which is a topological image of the total boundary* C *of* R. Γ *may consist of any finite or infinite number of Jordan curves, together with their limit points; or it may be* *some more general type of point-set. A definite sense of description is associated with each component of* Γ, *carried over from* C, *which may be supposed to be oriented so that* R *is on the left.*

To determine the existence of a minimal surface M *topologically equivalent to* R *and bounded by* Γ.

2. The exact analytic meaning of the requirement expressed in the last sentence is the same as in art. 4 of Note II.

If R has only a finite number k of boundaries and finite topological characteristic r, then we have the problem dealt with in Note II. This most important case of Problem P we shall call Problem P_0.

In Problem P_0, R is the Riemann semi-surface of a real *algebraic* curve; in Problem P, R is the Riemann semi-surface of a real *analytic* curve $\mathfrak{A}$. This means the manifold that is obtained from the Riemann surface $\mathfrak{R}$ associated with the equation $y = f(x)$ of the analytic curve $\mathfrak{A}$ by combining into a single geometric element each pair of conjugate complex points (x, y), $(\bar{x}, \bar{y})$ of $\mathfrak{A}$. That $\mathfrak{A}$ is *real* means that the power-series elements of the analytic function $y = f(x)$ occur in conjugate complex pairs, i.e., with conjugate complex centers x_0, $\bar{x}_0$ and coefficients a_m, $\bar{a}_m$.

The *boundary* C of R consists of the *real branches* of $\mathfrak{A}$, and R is two- or one-sided according as C does or does not separate $\mathfrak{R}$.

Thus, if $\mathfrak{A}$ is the curve

$$y^2 = \sin x \tag{1}$$

we have the problem of a minimal surface M topologically equivalent to a sphere with an infinite number of perforations that converge to a single point p, and where the prescribed boundary Γ of M consists of an infinite number of Jordan curves in space which converge to a single point P.

If, as another example, $\mathfrak{A}$ is

$$y^2 = (x^2 + 1) \sin x, \tag{2}$$

then the form required for M is that of a Möbius surface with an infinite number of boundaries, which have assigned positions Γ in space, of the same type of distribution as before.

If $\mathfrak{A}$ is

$$y^2 = \sin x \cosh x, \tag{3}$$

then the form of M is that of a one-sided surface with an infinite number of handles or branch-cuts converging to a point and an infinite number of given boundaries Γ converging to a point.

3. We introduce the following notation and definitions, and quote also the simple preliminary theorems (7°), (12°).

(1°) $\mathfrak{g}$ shall denote any representation of the given complete boundary Γ as topological image of the total boundary C of R.

(2°) *Definition.*

$$A(\mathfrak{g}, R) = \frac{1}{2}\int\int_R (\mathbf{H}_u^2 + \mathbf{H}_v^2)\, du\, dv, \tag{4}$$

where $\mathbf{H}(u, v)$ is the harmonic vector function on R whose boundary values are $\mathfrak{g}$.

(3°) $\mathfrak{T}$ shall denote the class of all riemannian manifolds topologically equivalent to R.

(4°) $\mathfrak{T}'$ shall denote all riemannian manifolds R' which are the limit of a sequence of manifolds R_m of $\mathfrak{T}$, without belonging to $\mathfrak{T}$ themselves.

Any such manifold R' either consists of a finite or infinite number of separate parts, or, if consisting of a single component, is of lower topological type than R. If the equation of the analytic curve $\mathfrak{A}'$ corresponding to R' is $F(x, y) = 0$, then, accordingly, either $F(x, y)$ is reducible, separating into a finite or infinite number of factors, or else it acquires new conjugate complex multiple points by coalescence of branch-points of R.

$\mathfrak{T}'$ may be regarded as the "frontier" of $\mathfrak{T}$ in the "space" of all riemannian manifolds.

(5°) *Definition.*

$$d(\Gamma, \mathfrak{T}) = \min A(\mathfrak{g}, R), \tag{5}$$

for all riemannian manifolds R belonging to $\mathfrak{T}$, and all topological representations $\mathfrak{g}$ of Γ as image of the boundary C of R. Throughout, we understand "min" in the sense of *lower bound,* without prejudice of the question as to whether the minimum is attained or not.

The value of d $(\Gamma, \mathfrak{T})$ *may be either a finite positive number, or* $+\infty$, depending on the case.

(6°) *Definition.*

$$d(\Gamma, \mathfrak{T}') = \min A(\mathfrak{g}, R'), \tag{6}$$

for all manifolds R' of $\mathfrak{T}'$, and all topological representations $\mathfrak{g}$ of Γ as image of the total boundary C of all components of R'.

(7°) THEOREM. *In every case,*

$$d(\Gamma, \mathfrak{T}) \leqq d(\Gamma, \mathfrak{T}'). \tag{7}$$

(8°) *Definition. For* $d(\Gamma, \mathfrak{T})$ *finite,*

$$e(\Gamma, \mathfrak{T}) = d(\Gamma, \mathfrak{T}') - d(\Gamma, \mathfrak{T}). \tag{8}$$

By the relation (7),

$$e(\Gamma, \mathfrak{T}) \geqq 0. \tag{9}$$

(9°) *Definition. Whether* d$(\Gamma, \mathfrak{T})$ *is finite or infinite,*

$$\bar{e}(\Gamma, \mathfrak{T}) = \max \overline{\lim_{m \to \infty}}\, e\,(\Gamma_m, \mathfrak{T}_m), \tag{10}$$

where Γ_m tends to Γ and $\mathfrak{T}_m$ to $\mathfrak{T}$ in such a way that $d(\Gamma_m, \mathfrak{T}_m)$ is finite for $m = 1, 2, 3, \ldots$. Sequences of this kind always exist for any given $(\Gamma, \mathfrak{T})$, since we may take $\mathfrak{T}_m$ to have always a finite number of boundaries and finite genus, and Γ_m to consist of a finite number of polygons. The maximum in (10) is with respect to all such sequences. "$\overline{\lim}$" denotes the superior limit.

Obviously, by (9), we have in every case,

$$\bar{e}(\Gamma, \mathfrak{T}) \geqq 0. \tag{11}$$

(10°) *Definition.*

$$a(\Gamma, \mathfrak{T}) = \min \mathfrak{A}(S) \tag{12}$$

where $\mathfrak{A}(S)$ denotes the area of S, which ranges over all surfaces of topological type $\mathfrak{T}$ bounded by Γ.

(11°) *Definition.*

$$a(\Gamma, \mathfrak{T}') = \min \mathfrak{A}(S') \tag{13}$$

where the range of S' consists of all surfaces of type $\mathfrak{T}'$ bounded by Γ.

(12°) THEOREM.

$$a(\Gamma, \mathfrak{T}) = d(\Gamma, \mathfrak{T}), \qquad \textit{also } a(\Gamma, \mathfrak{T}') = d(\Gamma, \mathfrak{T}'). \tag{14}$$

4. We are now in a position to state our main theorems.

THEOREM 1. *If* $d(\Gamma, \mathfrak{T})$ *is finite, and we have the strict form of inequality*[3]

$$d(\Gamma, \mathfrak{T}) < d(\Gamma, \mathfrak{T}'), \tag{15}$$

then a minimal surface M *exists, of topological type* T, *bounded by* Γ.

The area of M *is the least possible for its prescribed topological type and boundaries:*

$$\mathfrak{A}(M) = a(\Gamma, \mathfrak{T}). \tag{16}$$

THEOREM 2. *Whether* $d(\Gamma, T)$ *is finite or infinite, if* $\bar{e}(\Gamma, T)$*—always non-negative*[4]*—is actually positive:*

$$\bar{e}(\Gamma, \mathfrak{T}) > 0, \tag{17}$$

then a minimal surface M *exists, of topological type* $\mathfrak{T}$, *bounded by* Γ.

If $d(\Gamma, \mathfrak{T}) = a(\Gamma, \mathfrak{T})$ *is infinite, then so is the area of* M, *but every completely interior sub-region* M_1 *of* M *has an area which is finite and a minimum for its own topological type* $\mathfrak{T}_1$ *and boundaries* Γ_1.

5. The proof of these theorems is practically the same as for the finite or algebraic case of Problem P_0, requiring hardly more than the appropriate changes of wording. The same basic formulas (14), (15), (16) of Note II, are applied.

The important observation is that *the Variational Theorem of Note II (art. 7) continues to hold,* mutatis mutandis, *for all real analytic as well as algebraic curves* $\mathfrak{A}$.

We employ non-homogeneous line coördinates, namely the coefficients u, v in

$$y = ux + v. \tag{18}$$

We also take the point O in the statement of the Variational Theorem for origin, so that its line equation is $v = 0$. Then the line equation of the real analytic curve $\mathbf{A}$ will have the form

$$(au^2 + bu + c)\,K(u) + vL(u, v) = 0, \tag{19}$$

where K, L are now *any analytic functions*, instead of polynomials, as in Note II. The quadratic factor

$$au^2 + bu + c = 0 \tag{20}$$

represents the two conjugate imaginary tangents t, $\bar{t}$. For the varied equation, we have

$$[(a + a'\epsilon)u^2 + (b + b'\epsilon)u + (c + c'\epsilon)]\,K(u) + vL(u, v) = 0, \tag{21}$$

quite analogous to Note II.

6. The detailed version of the present note will be published elsewhere under the same title.

[1] In accordance with certain two simple postulates, which express essentially that the angle-metric induced on R by this conformal representation of its neighborhoods on a circle is self-consistent whenever two neighborhoods overlap. Cf. H. Weyl, *Die Idee der Riemannschen Fläche*, 36 (1913); L. Ahlfors, "Geometrie der Riemannschen Flächen," *Proc. Oslo Congress*, **1**, 239–248 (1936); S. Stoilow, "Sur la définition des surfaces de Riemann," *Ibid.*, **2**, 143; S. Stoilow, *Principes topologiques des fonctions analytiques*, (1937).

[2] This does not contradict Weyl, loc. cit., p. 66: "*Jede Riemannsche Fläche ist zweiseitig.*" For this theorem is associated with the idea of an analytic function $f(t)$ on the Riemann surface, t being sharply distinguished from its conjugate $\bar{t}$, whereas our geometric form of definition obviously permits one-sidedness of R. In that case, we employ in our analysis a covering surface R_1, which is two-sided and in two-one correspondence with R; cf. Weyl, loc. cit., last lines of p. 61.

[3] See (7).

[4] See (11).