*it is found that four parts of a trihorn have to be given in order to determine the remaining five parts.* This is entirely different from ordinary euclidean geometry and trihornometry of first order contact,[5] where only *three* parts are necessary to determine the remaining *three* parts.

[1] Kasner and Comenetz, "Conformal Geometry of Horn Angles," *Proc. Nat. Acad. Sci.*, **22**, No. 5, 303–309 (1936); Kasner, "Fundamental Theorems of Trihornometry," *Sci.*, **85**, No. 2211, 480–482 (1937); Kasner, "Trihornometry: A New Chapter of Conformal Geometry," *Proc. Nat. Acad. Sci.*, **23**, No. 6, 337–341 (1937).

[2] Kasner, "Conformal Geometry," *Proc. Fifth Internat. Congr. Math.*, Cambridge, **2**, 81 (1912); Kasner, "The Two Conformal Invariants of Fifth Order," *Trans. Amer. Math. Soc.* (1938) and "Schwarzian Symmetry," *Annals of Math.*, 1938.

[3] This is contrasted with a horn angle of first order which possesses the unique absolute conformal invariant of the third order $\frac{(x_2 - x_1)^2}{y_2 - y_1}$, where $x$ denotes the curvature and $y$ denotes the rate of variation of curvature at the given point.

[4] For horn angles of first order, we obtain an associated Finsler plane with the special Finsler metric $ds = \frac{dx^2}{dy}$. See Comenetz, "Kasner's Invariant and Trihornometry," *Amer. Math. Monthly*, **45**, 81–87 (1938). This metric is probably the *simplest conceivable example of a Finsler space* which is not merely riemannian.

[5] Kasner, "Trihornometry, a New Chapter in Conformal Geometry," loc. cit.

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# *A SKELETON LIFE TABLE*

By Carl R. Doering and Alice L. Forbes

Harvard School of Public Health, Department of Vital Statistics

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Life tables are being used considerably in vital statistics, notably by Raymond Pearl, Louis Dublin and Alfred Lotka. The accepted method of computation of life tables is the careful actuarial method used by the United States Bureau of the Census, and the Metropolitan Life Insurance Company, but many short methods of computation have been proposed with the idea of getting more quickly and easily a good idea of the expectations and of other life table functions at various ages: such are Yule's short method involving exponentials[1] and King's abridged method.[2] However, these so-called short methods still involve long computations, and often call for preliminary smoothing of the original data, which are usually too time-consuming to permit a health officer to apply the technique to local data. The method presented here seems to give very good results with remarkably few age groups, and in addition is very simple to compute.

We start with the age specific death rates, and all the rest of the work is derived from these. Let $m_x^{x+h}$ be the per capita death rate for the age group $x$ to $x + h$ years. We choose arbitrarily a value $l_0$, the number alive at age zero in a cohort. Then we compute $L_0^{0+h}$, the stationary population in that group; $d_0^{0+h}$, the deaths in the cohort during these years; and $l_{0+h}$ the survivors at age $0 + h$; and repeat the process throughout the table. In all intervals except the first and last we take

$$L_x^{x+h} = \frac{l_x}{\frac{1}{h} + \frac{1}{2} m_x^{x+h}}; \quad d_x^{x+h} = m_x^{x+h} L_x^{x+h}; \quad l_{x+h} = l_x - d_x^{x+h}.$$

The $\frac{1}{2}$ comes from an assumption that the deaths in an interval are spread evenly throughout the interval. Between the ages of 5 and 75 this is good enough, but in other cases it is not, and when it is not, we use

$$L_x^{x+h} = \frac{l_x}{\frac{1}{h} + (1 - c)m_x^{x+h}}$$

where $c$ is the fraction of the interval lived by those who die within the interval. This is most often used in the age group "under 5."

If the first age group is "under 1," the same formula can be used, but populations under one are apt to be poorly reported, giving too high death rates, so a simple alternative is employed. The births are used instead of the enumerated population, which divided into the deaths under one give the infant mortality. Consequently

$$d_0^1 = l_0 \times \text{I. M. (per capita), and } l_1 = l_0 - d_0^1$$

so we can proceed. But we shall need $L_0^1$ and that is equal to $l_1 + cd_0^1$.

In the last age group, it is obvious that $d_x^\infty = l_x$, so it is clear that

$$L_x^\infty = \frac{l_x}{m_x^\infty}$$

if we assume that the population from age $x$ on can be regarded as stationary. This enables one to fill in the table. The necessary values of $c$ are obtained from a study of the distribution of deaths by months for under one, or by single years for under five, but once determined for a given year and general area, need not be recomputed. For example, if we want to compute a life table for a county of New York State in 1930, we can get a good estimate of $c$ from the average age of deaths of infants or small children for the total New York State, and then apply it to our county, and it will be good enough.

With the columns $l_x$, $L_x^{x+h}$, $d_x^{x+h}$ filled in, we form a column $T_x$, for the populations above age $x$ by summing all $L_x^{x+h}$ from the bottom up, and then the expectation $\mathring{e}_x = T_x/l_x$.

A numerical example may make this clearer (see table 1).

TABLE 1

NEW YORK STATE, ONONDAGA COUNTY, 1929–31, DEATHS IN INSTITUTIONS REMOVED

| AGE | CENSUS POPUL. 1930 | AVERAGE DEATHS 1929–31 | $m_x^{x+h}$ | $l_x$ | $L_x^{x+h}$ | $d_x^{x+h}$ | $T_x$ | $\mathring{e}_x$ |
|---|---|---|---|---|---|---|---|---|
| 0–1 | 4,950* | 279.67 | 0.0565† | 1000 | 953‡ | 56 | 61,322 | 61.32 |
| 1–4 | 18,835 | 75.00 | 0.00398 | 944 | 3,744 | 15 | 60,369 | 63.98 |
| 5–9 | 26,138 | 42.33 | 0.00162 | 929 | 4,624 | 7 | 56,625 | 60.98 |
| 10–14 | 25,761 | 34.00 | 0.00132 | 921 | 4,590 | 6 | 52,000 | 56.45 |
| 15–19 | 23,557 | 47.67 | 0.00202 | 915 | 4,552 | 9 | 47,410 | 51.81 |
| 20–24 | 23,842 | 56.00 | 0.00235 | 906 | 4,503 | 11 | 42,858 | 47.31 |
| 25–29 | 22,712 | 64.00 | 0.00282 | 895 | 4,445 | 13 | 38,355 | 42.84 |
| 30–34 | 23,203 | 72.33 | 0.00312 | 883 | 4,379 | 14 | 33,910 | 38.42 |
| 35–44 | 45,593 | 259.00 | 0.00568 | 869 | 8,451 | 48 | 29,531 | 33.98 |
| 45–54 | 34,559 | 407.67 | 0.0118 | 821 | 7,753 | 91 | 21,080 | 25.67 |
| 55–64 | 23,823 | 594.33 | 0.0249 | 730 | 6,487 | 162 | 13,327 | 18.27 |
| 65–74 | 13,538 | 721.00 | 0.0533 | 568 | 4,484 | 239 | 6,840 | 12.05 |
| 75+ | 5,348 | 746.67 | 0.140 | 329 | 2,356 | 329 | 2,356 | 7.16 |
| | 291,859 | 3399.67 | | | 61,322 | 1000 | | |

* Average births for 1929–31. † Infant mortality. ‡ $c = .18$

NOTE: These figures are cut down from the original computation in which more places were carried.

We started with a radix of 1000. Then $1000 \times .0565 = 56$ deaths. $1000 - 56 = 994$. To get $L_0^1$, we took $944 + .18 \times 56 = 953$. From here on we proceeded by the regular formula, with $L_1^4 = \dfrac{944}{.25 + .00199} = 3744$; $d_1^4 = 3744 \times .00398 = 15$; and $l_5 = 944 - 15 = 929$; and so on. When we got $l_{75} = 329$, we immediately wrote $d_{75}^{\infty} = 329$, and then $L_{75}^{\infty} = \dfrac{329}{.140} = 2356$. $T_{75} = 2356$. $T_{65} = 2356 + 4484 = 6840$. $T_{55} = 6840 + 6487 = 13{,}327$, and so on, adding from the bottom up. $\mathring{e}_0 = \dfrac{61{,}322}{1000} = 61.32$. $\mathring{e}_1 = \dfrac{60{,}369}{944} = 63.98$. The sum of the $d_x^{x+h}$ column must equal $l_0$, and the sum of the $L_x^{x+h}$ must equal $T_0$.

The question now arises as to how good the results are—how close do they come to the results of the detailed actuarial methods? The published life tables of the United States Census for 1910 and 1920 give the raw data from which they were computed, so we can compare our work with these tables, and also with other short methods. Because King's method is constructed for five year intervals, and Yule feels that his is valid only for five year intervals, we shall use these small age groups first, combining into larger groups later. The published figures for 1910 are computed by single years, from which we have extracted the values comparable to ours (see table 2).

TABLE 2

PHILADELPHIA FEMALES 1910. COMPARISON OF THREE METHODS

| AGE | PUBLISHED $\mathring{e}_x$ | OUR METHOD $\mathring{e}_x$ | OUR METHOD ERROR | YULE'S METHOD $\mathring{e}_x$ | YULE'S METHOD ERROR | AGE | PUBLISHED $\mathring{e}_x$ | KING'S METHOD[3] $\mathring{e}_x$ | KING'S METHOD[3] ERROR |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 49.60 | 49.51 | −0.09 | 49.87 | 0.27 | | | | |
| 5 | 55.14 | 55.16 | 0.02 | 55.12 | −0.02 | 7 | 53.75 | 53.75 | 0.00 |
| 10 | 51.24 | 51.23 | −0.01 | 51.19 | −0.05 | 12 | 49.48 | 49.51 | 0.03 |
| 15 | 46.83 | 46.84 | 0.01 | 46.79 | −0.04 | 17 | 45.10 | 45.16 | 0.06 |
| 20 | 42.61 | 42.63 | 0.02 | 42.59 | −0.02 | 22 | 41.03 | 41.08 | 0.05 |
| 25 | 38.72 | 38.74 | 0.02 | 38.69 | −0.03 | 27 | 37.19 | 37.23 | 0.04 |
| 30 | 34.91 | 34.93 | 0.02 | 34.88 | −0.03 | 32 | 33.39 | 33.45 | 0.06 |
| 35 | 31.15 | 31.18 | 0.03 | 31.13 | −0.02 | 37 | 29.69 | 29.74 | 0.05 |
| 40 | 27.49 | 27.52 | 0.03 | 27.47 | −0.02 | 42 | 26.02 | 26.08 | 0.06 |
| 45 | 23.82 | 23.85 | 0.03 | 23.80 | −0.02 | 47 | 22.37 | 22.43 | 0.06 |
| 50 | 20.24 | 20.27 | 0.03 | 20.21 | −0.03 | 52 | 18.87 | 18.93 | 0.06 |
| 55 | 16.89 | 16.93 | 0.04 | 16.87 | −0.02 | 57 | 15.63 | 15.72 | 0.09 |
| 60 | 13.88 | 13.92 | 0.04 | 13.85 | −0.03 | 62 | 12.79 | 12.86 | 0.07 |
| 65 | 11.25 | 11.29 | 0.04 | 11.22 | −0.03 | 67 | 10.30 | 10.40 | 0.10 |
| 70 | 8.98 | 9.03 | 0.05 | 8.96 | −0.02 | 72 | 8.13 | 8.21 | 0.08 |
| 75 | 6.93 | 7.03 | 0.10 | 6.90 | −0.03 | 77 | 6.21 | 6.31 | 0.10 |
| 80 | 5.23 | 5.30 | 0.07 | 5.22 | −0.01 | 82 | 4.64 | 4.74 | 0.10 |
| 85 | 3.81 | 3.91 | 0.10 | 3.84 | 0.03 | 87 | 3.32 | 3.43 | 0.11 |
| 90 | 2.69 | 2.80 | 0.11 | 2.81 | 0.12 | 92 | 2.34 | 2.29 | −0.05 |
| 95 | 1.90 | 2.92 | 1.02 | 2.92 | 1.02 | 97 | 1.65 | 0.88 | −0.77 |
| 100 | 1.32 | 3.00 | 1.68 | 3.00 | 1.68 | | | | |

The marked error in the last two expectations in Yule's and our methods is accounted for by the fact that the observed $m_x^{x+h}$'s were biologically impossible, with $m_{100}^{\infty} < m_{95}^{99} < m_{90}^{94}$. The same error does not appear in King's method since the latter provides for rejection of these values, and extrapolation to obtain substitute values. Except for these terminal entries, our method is as good as the others, and much more rapid. (Yule's method is almost as rapid if sufficiently extensive exponential tables had been available, but logarithms had to be used.)

In Whipple's "Vital Statistics" it is shown that seven groups for adjusting death rates do as well for practical purposes as eleven groups.[4] Presumably it should be possible to compute a skeleton life table including a figure for the expectation of life at birth on fewer age groups than we have just used without losing much accuracy and thereby to gain a great deal in ease of computation. Furthermore, in many cases of interest to the health officer it is impossible to get enumerated populations or reported deaths except in fairly broad age groups so that if a life table is to be computed directly from the available data it must be of skeleton form.

Now what does happen when we use fewer groups? There are countless types of groups possible, but we shall consider those which we think would be apt to be used because of ready availability.

First, take the 13 groups used by the United States Census in publishing populations for moderately large places. The infants are kept separate from the children 1 to 4; there are five year groups from 5 to 35, and 10 year groups from 35 to 75 and all 75 and over are lumped into a final group [see table 3*a*].

TABLE 3

PHILADELPHIA FEMALES 1910. COMPARISON OF THREE GROUPINGS

| AGE | PUBLISHED $\mathring{e}_x$ | 3*a* OUR METHOD $\mathring{e}_x$ | 3*a* OUR METHOD ERROR | 3*b* OUR METHOD $\mathring{e}_x$ | 3*b* OUR METHOD ERROR | 3*c* OUR METHOD $\mathring{e}_x$ | 3*c* OUR METHOD ERROR |
|---|---|---|---|---|---|---|---|
| 0 | 49.60 | 49.54 | −0.06 | 49.61 | +0.01 | 49.58 | −0.02 |
| 1 | 55.28 | 55.16 | −0.12 | | | 55.20 | −0.08 |
| 5 | 55.14 | 55.13 | −0.01 | 55.29 | +0.15 | 55.17 | +0.03 |
| 10 | 51.24 | 51.20 | −0.04 | | | | |
| 15 | 46.83 | 46.81 | −0.02 | 46.96 | +0.13 | | |
| 20 | 42.61 | 42.60 | −0.01 | | | 42.64 | +0.03 |
| 25 | 38.72 | 38.71 | −0.01 | 38.91 | +0.19 | | |
| 30 | 34.91 | 34.89 | −0.02 | | | | |
| 35 | 31.15 | 31.14 | −0.01 | | | | |
| 45 | 23.82 | 23.81 | −0.01 | 24.08 | +0.26 | 23.91 | +0.09 |
| 55 | 16.89 | 16.88 | −0.01 | | | | |
| 65 | 11.25 | 11.28 | +0.03 | 11.54 | +0.29 | 11.28 | +0.03 |
| 75 | 6.93 | 6.93 | 0.00 | | | 6.93 | 0.00 |

Our method fits slightly better than with the complete five year groups[5] of table 2, no matter which of the three methods is used, suggesting that perhaps the grouping acts somewhat as a smoothing of the $m_x^{x+h}$'s. The assumption of a stationary population over 75 is not strictly valid; it gives the exact answer here, but that occurs rarely, and is not to be expected. A number of such calculations for different populations shows that there may be considerable error.

Second, for smaller places, the census often prints populations in six age groups beginning with "under 5," then two ten year groups, two 20 year groups and finishing with 65+ (see table 3*b*). Although the errors here are noticeable, still the results are probably good enough for practical purposes.

Third, we may give another grouping, with only 7 age groups, which may be made up from the 13 used above and which seem to have a somewhat greater biological or health interest than the second grouping (see table 3*c*).

We have used the enumerated populations and the reported deaths in certain age groups without smoothing or other adjustment. It is interesting to inquire what would be the result of applying our method to the age specific death rates taken out of the published life table. How well will the life table be reproduced? In five year groups, or the 13 census groups the reproduction of the table is excellent, though not notably better than when based on raw data. When done in the 6 census groups or 7 biological groups the errors seem to be considerably greater than for the computation as made directly from reported data. For other populations (than Philadelphia Females 1910) the results are similar and lead to the same conclusions, namely:

The short method of computation based on 13 census groups or the 7 so-called biological groups seems to give an entirely adequate life table for those entries computed, and the 6 census groups used for the smaller places will still give a tolerably satisfactory skeleton life table. While without a wide experience it would be impossible to tell under what circumstances, if any, the method would prove to give seriously incorrect results, a large experience indicates that the method can be recommended to health officers as likely to give sufficiently good results for their purposes, and is so short that the calculation can be made readily enough to make the life table technique almost as simple as that of adjusting death rates.

[1] "Some Life Table Approximations," by G. Udney Yule. *Proc.* Internat. Math. Cong., held in Toronto, Aug. 11–16, 1924. Vol. II, p. 873.

[2] *Length of Life*, by Louis I. Dublin and Alfred J. Lotka, p. 312.

[3] The figures for King's method as given here were the result of applying his method to the raw data—not the smoothed data which he recommends. Following his method rigidly gives a very good fit to published data, but the method was used on raw data for comparability with the other methods, and because a health officer desiring a short method would certainly be applying it to raw data.

[4] *Vital Statistics*, by George C. Whipple, p. 297.

[5] A trial of Yule's method for the groupings of table 3 indicates that it does not give such good results as ours, besides being somewhat slower in computation. We shall therefore omit further comparisons with it.