SCATTERING OF SLOW NEUTRONS BY PROTONS

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An elementary discussion is given here of the scattering of slow neutrons by protons for the spherical well potential and for the meson type potential. Calculations have been carried out for the potential hole because many of the discussions of this problem have assumed that the interaction is of negligible width and this assumption leads to appreciable inaccuracies. The question of the introduction of the virtual level of the deuteron to describe the singlet scattering is also considered, and in this connection the sequence of virtual levels defined by the Kapur-Peierls[1] scattering procedure is determined to illustrate the convergence of the method in the extreme case of the deuteron.

1. For neutrons of low relative velocities only the states of zero angular momentum contribute appreciably to the scattering. For a potential hole of depth C and width z, the S-wave $\varphi = r\psi$ satisfies the wave equation

$$d^2\varphi/dx^2 + (b + a)\varphi = 0, \qquad x < x_0; \tag{1}$$

$$d^2\varphi/dx^2 + a\varphi = 0, \qquad x > x_0; \tag{1a}$$

where the unit of length is $r_0 = e^2/mc^2$, and $x = r/r_0$, $x_0 = z/r_0$, $C = (h^2/Mr_0^2)b = 10.24\ b\ mc^2$, $E = (h^2/Mr_0^2)a$.* The solution of equation (1) which vanishes at $x = 0$ is, for $a < 0$,

$$\varphi_i(x) = N \sin (b - \epsilon)^{1/2}\, x, \qquad x \leqq x_0; \tag{2}$$

$$\varphi_0(x) = C_1 \exp(-(\epsilon)^{1/2}(x - x_0)) + C_2 \exp((\epsilon)^{1/2}(x - x_0)), \qquad x \geqq x_0; \tag{3}$$

where $\epsilon = -a$. Continuity at $x = x_0$ in φ and φ' gives

$$C_1 = (N/2)\left\{\sin (b - \epsilon)^{1/2}x_0 - ((b - \epsilon)/\epsilon)^{1/2} \cos (b - \epsilon)^{1/2}x_0\right\}; \tag{4}$$

$$C_2 = (N/2)\left\{\sin (b - \epsilon)^{1/2}x_0 + ((b - \epsilon)/\epsilon)^{1/2} \cos (b - \epsilon)^{1/2}x_0\right\}. \tag{5}$$

For $a \geqq 0$, one has

$$\varphi_i(x) = N \sin (b + a)^{1/2}x, \qquad x \leqq x_0; \tag{6}$$

$$\varphi_0(x) = N\{\sin (b + a)^{1/2} \cos (a)^{1/2}(x - x_0) + (b + a)/a)^{1/2} \cos (b + a)^{1/2} \sin (a)^{1/2}(x - x_0)\}, \; x \geqq x_0. \tag{7}$$

In order that a stable state exist one must have $C_2 = 0$, and for $E(^3S) = -4.257\ mc^2$, $\epsilon(^3S) = 0.4157$, a relationship is determined by this condition between $b(^3S)$ and x_0 which gives $b(^3S) = 4.010$ if one takes $x_0 = 1.000$.

In the following it will be assumed that $x_0 = 1.000$ for both the triplet and the singlet interactions.

The scattering cross-section for zero energy may be written in the form

$$\sigma = \pi r_0^2 \left\{ [(\varphi(1)/\varphi'(1))_{^1S} - 1]^2 + 3[(\varphi(1)/\varphi'(1))_{^3S} - 1]^2 \right\}; \quad (8)$$

and one has further

$$(\varphi(1)/\varphi'(1))_{^1S} = \tan (b(^1S))^{1/2}/(b(^1S))^{1/2}; \quad (\varphi(1)/\varphi'(1))_{^3S} = \tan (b(^3S))^{1/2}/(b(^3S))^{1/2}. \quad (9)$$

Taking $\sigma = 18.3 \times 10^{-24}$ cm.^{2}, one finds $b(^1S) = 2.235$. If one keeps x_0 fixed, the determination of $b(^1S)$ is not appreciably affected by a change in the value of σ. For the scattering amplitudes one has the following numerical values

$$A(^1S) = \left\{\tan (b(^1S))^{1/2}/(b(^1S))^{1/2}\right\} - 1 = 7.804;$$
$$A(^3S) = \left\{\tan (b(^3S))^{1/2}/(b(^3S))^{1/2}\right\} - 1 = -2.084. \quad (10)$$

The following approximation has sometimes been used for the triplet scattering amplitude:

$$A(^3S) \approx -1/(\epsilon(^3S))^{1/2} = -1.551. \quad (11)$$

For a hole of width $x_0 = 1.000$ it is evident that this approximation gives an error in the scattering intensity of 50 per cent. A better approximation may be found[2] by solving the equation $C_2 = 0$ for $b(^3S)$ in terms of $\epsilon(^3S)$. This procedure gives

$$A(^3S) \approx - [(\epsilon(^3S))^{1/2} + \epsilon(^3S)/2]^{-1} - 1, \quad (12)$$

and this approximation is in error by 9 per cent for the scattering intensity. To get an expression similar to (12) one may attempt to introduce an energy characteristic of the singlet interaction in terms of which $A(^1S)$ may be expressed. It is readily seen that for $b = b(^1S) = 2.235$ there is no solution for $C_2 = 0$, and since there is no stable singlet level, a positive or virtual energy level has been introduced. One definition which has been given[3] of the virtual energy level is equivalent to the following: let ϵ' be the absolute value of a negative energy level for which $C_1 = 0$, then the positive energy level for which $a' = \epsilon'$ is the virtual energy level. For $b(^1S) = 2.235$, $a' = 0.0145$. One might also define the virtual energy as that positive energy a_0 for which

$$(\varphi'_{a_0}(1)/\varphi_{a_0}(1))_{^1S} = (a_0)^{1/2}. \quad (13)$$

$\varphi_{a_0}(x)$ then has a phase constant of $\pi/4$ as may be seen from (7). The value of a_0 determined by (13) corresponds to an energy of 61,000 e.-v. for $b(^1S) = 2.235$. An approximate solution of Eq. (13) gives

$$A(^1S) \approx \left\{(a_0)^{1/2} + a_0/2\right\}^{-1} - 1, \quad (14)$$

and this approximation is in error by less than 0.1 per cent.

2. The dependence of the potential on the neutron-proton distance as derived from the meson field theory is of the form $C\ [\exp\{-\lambda r/r_0\}/\lambda(r/r_0)]$. The S-wave $\varphi = r\psi$ satisfies the equation

$$d^2\varphi/dx^2 + (be^{-x}/x)\varphi + a\varphi = 0, \quad (15)$$

where $x = \lambda r/r_0$, $b = CMr_0^2/h^2\lambda^2$, $a = EMr_0^2/h^2\lambda^2$.

For thermal neutrons ($a = 0$) the scattering amplitude is

$$A = \lim_{x \to \infty} \left\{\varphi(x)/\varphi'(x) - x\right\},$$

and the scattering cross-section is given by

$$\sigma = (\pi r_0^2/\lambda^2)\left\{A(^1S)^2 + 3A(^3S)^2\right\}. \quad (16)$$

The scattering amplitude has been determined by numerical integration of (15) for several values of b, and table 1 summarizes the results. For a value b_2 near a value b_1 in this table one may calculate $\varphi_2'(x)/\varphi_2(x)$ and

TABLE 1

b	1.440	1.500	1.516	1.520	2.550	2.600	2.700
A	8.452	11.64	12.87	13.22	−3.229	−3.065	−2.775

hence the scattering amplitude, by the following perturbation formula:

$$\varphi_2'(x)/\varphi_2(x) - \varphi_1'(x)/\varphi_1(x) = (b_1 - b_2)\left[\int_0^x \varphi_1\varphi_2(e^{-x}/x)dx/\varphi_1(x)\varphi_2(x)\right] \approx (b_1 - b_2)\left[\int_0^x \varphi_1^2(e^{-x}/x)dx/\varphi_1^2(x)\right]. \quad (17)$$

It may be of interest to note that for $b = 1.505$, which is the value for the singlet interaction, approximately one-half of the scattered intensity comes from the region beyond $r = 2r_0$; and beyond $r = 6r_0$ the scattered intensity remains essentially constant. For $b = 2.600$ which is the value for the triplet interaction the scattered intensity remains practically constant beyond $r = 2r_0$.

The values given above for $b(^1S)$ and $b(^3S)$ have been determined in the following way: for an assumed value of $b(^3S)$ the corresponding value of $\lambda(^3S)$ is determined from the binding energy of H^2; if one assumes $\lambda(^3S) = \lambda(^1S)$, then $b(^1S)$ is determined from the scattering cross-section (18.3×10^{-24} cm.^{2}) for thermal neutrons using table 1; the binding energy of H^3 is then used to fix the correct value of $b(^3S)$. This procedure gives the first column of table 2. If σ were taken to be 13.1×10^{-24} cm.^{2}, one gets the values given in the third column of table 2. If one does not assume $\lambda(^3S) = \lambda(^1S)$; but, for example, takes $\lambda(^1S) = 2.38$,[4] a value which fits proton-proton scattering data, then the other constants determined by the above procedure are given in the second column of the table.

TABLE 2

σ (CM.²)	18.3×10^{-24}		13.1×10^{-24}
$b(^3S)$	2.600	2.42	2.54
$b(^1S)$	1.505	1.56	1.48
$C(^3S)(mc.^2)$	62.80	87.6	70.1
$C(^1S)(mc.^2)$	36.35	90.5	40.8
$\lambda(^3S)$	1.536	1.88	1.64
$\lambda(^1S)$	1.536	2.38	1.64

3. In their derivation of the nuclear dispersion formula Kapur and Peierls[1] define the virtual levels which determine nuclear scattering by means of a boundary condition at some R which is a distance such that $V(r)$ is negligible for $r \geqq R$. In the case of neutron-proton scattering this procedure may be carried through accurately and gives an alternative to the usual scattering calculation. For thermal neutrons ($a = 0$) the sequence of virtual levels is determined by the boundary condition $\varphi'(R) = 0$. In every case the sequence of virtual levels will depend on the value taken for R.

For the potential hole of width r_0 the virtual energy values are given by

$$a_n = (2n + 1)^2(\pi^2/4) - b; \qquad (n = 0, 1, 2 \ldots) \qquad (18)$$

where R also is taken to be r_0 and the incident neutron energy is taken to be zero. Table 3 gives the singlet and triplet virtual energy sequences. The scattering cross-section is now given by

$$\sigma = \pi r_0^2\{(\sum_n 2/a_n(^1S) - 1)^2 + 3(\sum_n 2/a_n(^3S) - 1)^2\}. \qquad (19)$$

TABLE 3

n	0	1	2	3
$a_n(^1S)$	0.2324	19.97	59.45	118.7
$a_n(^3S)$	−1.543	18.20	57.67	116.9

In the case of the meson potential it is convenient to take $R = 4.732\ r_0$; an appreciably larger value for R gives slow convergence in the contributions from the higher levels while an appreciably smaller value does not give the proper scattering since the interaction is then large at the boundary. The scattering cross-section is given by (16) where the scattering amplitudes now have the form

$$A = \sum_n \varphi_n^2(X)/a_nN_n - X, \qquad (20)$$

where $\varphi_n(X)$ is the value of the wave function for the nth virtual level at $X = \lambda R/r_0$ and $N_n = \int_0^X \varphi_n^2 dx$. The virtual energy values of the first few levels, together with the values of $\varphi_n^2(X)/N_n$, are given in table 4. These values may be compared with the virtual energy levels obtained for the potential hole of width r_0 when R is also taken to be $4.732\ r_0$: $a_n(^1S) =$ 0.0242, 0.603, 2.23; $a_n(^3S) = -0.419$, 0.291, 1.91.

TABLE 4

n	0	1	2
$\lambda^2 a_n(^1S)$	0.02522	0.6372	...
$\lambda^2 a_n(^3S)$	−0.4215	0.2720	1.77
$(\varphi_n^2/N_n)_{^1S}$	0.1845	0.2901	...
$(\varphi_n^2/N_n)_{^3S}$	0.01736	0.3791	0.3067

As one would expect, the convergence for $R = 4.732\ r_0$ is not so rapid as was the case for the potential hole where R could be taken to be r_0. Even for the latter case the first three levels should be taken to get an accurate result for the scattering amplitude.

* $\hbar$ denotes Planck's constant divided by 2π.

[1] Kapur, P. R., and Peierls, R., *Proc. Roy. Soc.*, **166**, 277 (1938).
[2] Wigner, E., *Zeit. Physik*, **83**, 253 (1933).
[3] Breit, G., Thaxton, H. M., and Eisenbud, L., *Phys. Rev.*, **55**, 1018 (1939).
[4] Share, S. S., Hoisington, L. E., and Breit, G., *Ibid.*, **55**, 1130 (1939).

ON THE FORMATION OF CLUSTERS OF NEBULAE AND THE COSMOLOGICAL TIME SCALE

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A. The Problem of the Short Time Scale in an Expanding Universe.—The red-shift in the spectra of nebulae is most easily interpreted as a Doppler effect caused by the mutual and approximately uniform recession of these nebulae from one another. This hypothesis of an expanding universe leads to an alarmingly short cosmological time scale with the consequence that some two billion years ago intergalactic space must have been practically non-existent and the nebulae were not clearly separable from one another. Nevertheless, the individual objects which constitute these nebulae such as the stars, and the double, triple and multiple systems of stars might well have existed in their present physical condition even in a very contracted universe. The argument that the earth and the stars have existed for periods longer than two billion years, and the fact that perhaps the present statistical distribution of stars could have been achieved only in a very much longer time than two billion years is therefore not sufficient to reject the hypothesis of the expanding universe. In order to show that the time scale demanded by this hypothesis is inacceptable we must endeavor to discuss objects which (a) in a contracted universe clearly could not have existed in their present form and (b) whose formation required intervals of time definitely larger than two billion years. If we