# SOLUTION OF THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS

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1. *Introduction.*—One of the most important hitherto unsolved problems of the calculus of variations is the so-called "inverse problem," namely: *Given any family of* $\infty^{2n}$ *curves (paths) in* $(n + 1)$*-dimensional space* $(x, y_j)$, $(j = 1, \ldots, n)$, *as represented by a system of differential equations*

$$y_i'' = F_i(x, y_j, y_j') \qquad (i = 1, \ldots, n); \tag{1.1}$$

*to determine whether these curves can be identified as the totality of extremals of a variation problem*

$$\int \varphi(x, y_j, y_j')dx = \text{min.}, \tag{1.2}$$

*and in the affirmative case to find all the corresponding functions* $\varphi$.

The author has recently been able to find a complete solution of this problem. The purpose of this note is to announce and to present the essential features of this solution. A fully detailed account has already been prepared and will appear in the *Annals of Mathematics*.

2. Important progress toward the solution of the inverse problem was made by D. R. Davis in a Chicago dissertation of about a decade ago.[1] Following the work of Hirsch and Kürschak in other cases of the inverse problem,[2] he proved that the self-adjointness of the variational system, $\delta\omega_i$, of the "Euler expressions,"

$$\omega_i \equiv \frac{\partial\varphi}{\partial y_i} - \frac{d}{dx}\frac{\partial\varphi}{\partial y_i'} \qquad (i = 1, \ldots, n), \tag{2.1}$$

is a sufficient as well as necessary condition for distinguishing the Euler systems among those of the general form $\omega_i(x, y_j, y_j', y_j'')$. The problem is thus transferred to the determination of multipliers $P_{ij}(x, y_k, y_k')$ of the equations (1.1) such that the variational system, $\delta E_i$, of the differential expressions

$$E_i = P_{ij}(F_j - y_j'') \tag{2.2}$$

is self-adjoint.[3]

Davis finds that these multipliers $P_{ij}$ are the solutions of a certain linear differential system. Difficulties arise, however, in attempting to solve this system in the general case, and Davis contents himself with the consideration of certain three particular examples.

3. The same differential system,[4] called $S$, occurs in our work, but is obtained by a different method and interpreted from a different point of view. We arrive at $S$ as a consequence of certain identities (5.1, 5.2) concerning the Euler expressions $\omega_i$. To be sure, in the last analysis, these identities are but expressive of the self-adjoint nature of $\delta\omega_i$; nevertheless this interpretation is not relevant to our purpose, and indeed the whole idea of self-adjointness plays no rôle in our theory.

Our essential contribution consists in the *complete solution of the system S*, resulting particularly in a *general method for distinguishing extremal and non-extremal families*. As a special application, we give incidentally, in (6.2), (6.3), *the first actual examples of non-extremal families*.

By solution of $S$, we mean, of course, an existence-theoretic one, i.e., the determination of the consistency of the system and of the arbitrary constants or functions involved in its general integral.

In §§7, 8 a *tensor form of solution* is presented, employing parametric representation and covariant differentiation with respect to the given space of paths.

4. The extremals of (1.2) are defined by the Euler-Lagrange equations

$$\frac{\partial \varphi}{\partial y_i} - \frac{d}{dx}\frac{\partial \varphi}{\partial y'_i} = 0 \qquad (i = 1, \ldots, n), \tag{4.1}$$

where the operator $d/dx$ denotes total differentiation along a path:

$$\frac{d}{dx} \equiv \frac{\partial}{\partial x} + y'_j \frac{\partial}{\partial y_j} + F_j \frac{\partial}{\partial y'_j}. \tag{4.2}$$

Analytically expressed, our inverse problem consists then in the solution for $\varphi$, as unknown function, of the differential system

$$\frac{\partial \varphi}{\partial y_i} - \left(\frac{\partial}{\partial x} + y'_j \frac{\partial}{\partial y_j} + F_j \frac{\partial}{\partial y'_j}\right)\frac{\partial \varphi}{\partial y'_i} = 0 \qquad (i = 1, \ldots, n), \tag{4.3}$$

the functions $F_j$ being given. Further, these equations must be solvable as a linear algebraic system for $F_j$ or $y''_j$, in order that the differential equations of the extremals may have the prescribed form (1.1). Therefore a solution of (4.3) is required such that

$$\operatorname{Det} |\varphi_{ij}| \neq 0, \tag{4.4}$$

where

$$\varphi_{ij} \equiv \frac{\partial^2 \varphi}{\partial y'_i \partial y'_j}. \tag{4.5}$$

The determinant (4.4) is, of course, the *Hessian* of $\varphi$.

For the case $n = 1$, of a 2-dimensional containing space, the solution of the inverse problem has long been known and is given in the standard text books.[5] The system (4.3) then consists of but a single equation, whose solvability for $\varphi$ is assured by standard existence theorems.

The difficulty of the problem resides in the case $n \geqq 2$, i.e., for a 3- or higher dimensional space.

5. *The Fundamental Differential System S.*—As stated in the introduction, our method depends on replacing the Euler-Lagrange system (4.3) by a more easily manageable system $S$, where the unknowns are the $n(n + 1)/2$ functions $\varphi_{ij}$ defined by (4.5).

This transformation is based on certain identities which we establish, involving the Euler expressions (2.1), namely:

$$\frac{\partial \omega_i}{\partial y_j'} + \frac{\partial \omega_j}{\partial y_i'} + 2\frac{d}{dx}\varphi_{ij} + \frac{\partial F_k}{\partial y_j'}\varphi_{ik} + \frac{\partial F_k}{\partial y_i'}\varphi_{jk} \equiv 0, \tag{5.1}$$

$$\frac{d}{dx}\left(\frac{\partial \omega_i}{\partial y_j'} - \frac{\partial \omega_j}{\partial y_i'}\right) - 2\left(\frac{\partial \omega_i}{\partial y_j} - \frac{\partial \omega_j}{\partial y_i}\right) + \frac{1}{2}\frac{\partial F_k}{\partial y_i'}\left(\frac{\partial \omega_j}{\partial y_k'} + \frac{\partial \omega_k}{\partial y_j'}\right) - \frac{1}{2}\frac{\partial F_k}{\partial y_j'}\left(\frac{\partial \omega_i}{\partial y_k'} + \frac{\partial \omega_k}{\partial y_i'}\right) + A_{jk}\varphi_{ik} - A_{ik}\varphi_{jk} \equiv 0, \tag{5.2}$$

where

$$A_{jk} \equiv \frac{d}{dx}\frac{\partial F_k}{\partial y_j'} - 2\frac{\partial F_k}{\partial y_j} - \frac{1}{2}\frac{\partial F_m}{\partial y_j'}\frac{\partial F_k}{\partial y_m'}. \tag{5.3}$$

If the Euler-Lagrange equations are satisfied, i.e., $\omega_i = 0$, these identities imply the following system of equations for $\varphi_{ij}$:

$$\frac{d}{dx}\varphi_{ij} + \frac{1}{2}\frac{\partial F_k}{\partial y_j'}\varphi_{ik} + \frac{1}{2}\frac{\partial F_k}{\partial y_i'}\varphi_{jk} = 0, \tag{5.4}$$

$$A_{jk}\varphi_{ik} - A_{ik}\varphi_{jk} = 0. \tag{5.5}$$

We also have evidently by (4.5)

$$\frac{\partial \varphi_{ij}}{\partial y_k'} = \frac{\partial \varphi_{ik}}{\partial y_j'}, \tag{5.6}$$

and we take explicit notice of the symmetry of $\varphi_{ij}$ in its indices:

$$\varphi_{ij} = \varphi_{ji}. \tag{5.7}$$

The linear differential system consisting of the equations (5.4)–(5.7) and the inequation (4.4):

$$\text{Det} \left| \varphi_{ij} \right| \neq 0, \tag{5.8}$$

will be denoted by $S$.

Conversely, we prove the

PROPOSITION.[6] *To every solution $\varphi_{ij}$ of the system $S$ there corresponds a solution $\varphi$, with non-vanishing Hessian, of the Euler-Lagrange equations* (4.3), *$\varphi$ being related to $\varphi_{ij}$ by* (4.5). *For a given system $\varphi_{ij}$, the function $\varphi$ is uniquely determined except for the inevitable addition of an arbitrary exact differential to $\varphi dx$.*

The addition of the exact differential is inevitable because, as is obvious, this has no effect on the extremals.

6. In our detailed paper we have given a complete solution of the system $S$ by means of the standard theory of differential systems.[7]

This general method of solution is then applied to provide a full discussion of the most important and sufficiently typical case of 3 dimensions ($n = 2$), with enumeration of all the possible types. The classification is arranged to a large extent according to the rank of a certain matrix.

$$\begin{Vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{Vmatrix} \tag{6.1}$$

whose elements are known functions depending on the given curve family.

In order that a given family of $\infty^4$ curves in the 3-space $(x, y, z)$, as defined by differential equations

$$y'' = F(x, y, z, y', z'), \qquad z'' = G(x, y, z, y', z'),$$

be an extremal family, it is at least a necessary condition that *the determinant of the preceding matrix shall vanish.* By non-satisfaction of this condition, therefore, examples of non-extremal families can be constructed *ad libitum;* for instance,

$$y'' = y^2 + z^2, \qquad z'' = y \tag{6.2}$$

is a non-extremal family. Many others can be found by means of other necessary conditions occurring in our work, e.g.,

$$y'' = y^2 + z^2, \qquad z'' = 0. \tag{6.3}$$

7. *Tensor Form of Solution.*—We conclude our detailed paper with a tensor form of solution of the inverse problem, which has the advantage of greater symmetry and perspicuity of the formulas.

Parametric representation is employed throughout; we take the given system of paths in the form[8]

$$\frac{d^2x^i}{dt^2} = H^i(x^j, p^j), \quad \left(p^j \equiv \frac{dx^j}{dt}\right), \quad (i, j = 1, \ldots, n), \tag{7.1}$$

the functions $H^i$ being homogeneous of the second degree in the arguments $p^j$, while the variation problem is sought for in the form

$$\int \psi(x^j, p^j)dt = \min. \tag{7.2}$$

where $\psi$ is homogeneous of the first degree in the $p^j$.

Covariant differentiation based on the given space of paths is used instead of ordinary partial differentiation.[9] Covariant differentiation as to $x^c$ will be denoted by a comma followed by the letter $c$, and the operation $\partial/\partial p^k$ (also a covariant process) by a dot followed by $k$. We also define an operator $\delta$ signifying covariant total differentiation as to $t$ along a path:

$$\delta T^{(a)}_{(b)} \equiv p^c T^{(a)}_{(b),\,c}, \tag{7.3}$$

where $T^{(a)}_{(b)}$ denotes a tensor with any system $(a)$ of contravariant indices and $(b)$ of covariant indices.

The *Euler vector* of the scalar $\psi$ is, by definition,

$$\omega_i \equiv \psi_{,i} - \delta\psi_{.i}. \tag{7.4}$$

We find that this obeys the following identities (tensor form of (5.1), (5.2)):

$$\omega_{i.j} + \omega_{j.i} + 2\delta\psi_{ij} \equiv 0, \tag{7.5}$$

$$\omega_{i,\,j} - \omega_{j,} \quad - \frac{1}{2}\,\delta(\omega_{i.j} - \omega_{j.i}) + B^a_{**j}\psi_{ia} - B^a_{**i}\psi_{ja} \equiv 0. \tag{7.6}$$

Here

$$\psi_{ij} \equiv \psi_{.ij} \equiv \frac{\partial^2\psi}{\partial p^i \partial p^j},$$

while

$$B^a_{**j} = p^i p^k B^a_{ikj},$$

where $B^a_{ikj}$ denotes the curvature tensor of the given space of paths.

The Euler-Lagrange equations, $\omega_i = 0$, then give:

$$\delta\psi_{ij} = 0, \tag{7.7}$$

$$B^a_{**j}\psi_{ia} - B^a_{**i}\psi_{ja} = 0. \tag{7.8}$$

To these we adjoin the obvious equations:

$$\psi_{ij.k} = \psi_{ik.j}, \tag{7.9}$$

$$\psi_{ij} = \psi_{ji}, \tag{7.10}$$

$$p^a\psi_{ia} = 0. \tag{7.11}$$

The last follows by Euler's equation from the zero degree homogeneity of $\psi_{\cdot i}$, implied by the first degree homogeneity of $\psi$.

We must include also the condition

$$\text{Rank of } \| \psi_{ij} \| = n - 1. \tag{7.12}$$

*The system* (7.7)–(7.12), *for the tensor* $\psi_{ij}$ *as unknown, is the fundamental differential system S in the tensor form of solution.*

8. Among the consequences of the system $S$ we may notice

$$\psi_{ij,\,k} = \psi_{ik,\,j}, \tag{8.1}$$

and

$$B^{a}_{*jk}\psi_{ia} + B^{a}_{*ki}\psi_{ja} + B^{a}_{*ij}\psi_{ka} = 0, \tag{8.2}$$

where $B^{a}_{*jk} \equiv p^{i}B^{a}_{ijk}$. In fact, (8.2) implies (7.8) in the presence of the equations (7.11), so that (7.8) may be replaced by (8.2).

We can transform (8.2) into a form invariant under change of parameter on the paths with the use of the Weyl tensor $W^{i}_{jkl}$ or its related tensor

$$W^{i}_{*kl} \equiv p^{j}W^{i}_{jkl}.$$

In these terms, (8.2) becomes

$$W^{a}_{*jk}\psi_{ia} + W^{a}_{*ki}\psi_{ja} + W^{a}_{*ij}\psi_{ka} = 0. \tag{8.3}$$

By a discussion based on this equation, we prove as incidental result the following theorem:

*Every family of paths whose Weyl tensor vanishes is an extremal family.*

Another way of expressing this result is:

*Every space of paths whose Weyl tensor vanishes is identifiable with a Finsler space of paths.*

A Finsler space, as is well known, is one whose length-element is $ds = \psi(x, dx)$, this being the element of integration in (7.2). The paths of a Finsler space are its geodesics, the extremals of $\int \psi(x, dx) = \min$.

[1] D. R. Davis, "The Inverse Problem of the Calculus of Variations in Higher Space," *Trans. Amer. Math. Soc.*, **30**, 710–736 (1928). This paper deals with the 3-dimensional case. It was followed by "The Inverse Problem of the Calculus of Variations in a Space of $(n + 1)$ Dimensions," *Bull. Amer. Math. Soc.*, **35**, 371–380 (1929), by the same author. These papers will be referred to respectively as I, II.

[2] For historical remarks and references, see the introduction to II.

[3] In reference to (2.2), *the summation convention is applied to repeated indices throughout this note.*

[4] Except that Davis does not make explicit the rôle of the inequation (5.8) of our system $S$.

[5] See Bolza, *Lectures on the Calculus of Variations*, New York, 1931, pp. 31–32.

[6] Cf. I, pp. 717–719; II, p. 375, et seq.

[7] As presented, for example, in the treatises of C. Riquier, M. Janet and J. M. Thomas.

The discussion in Janet, *Leçons sur les Systèmes d'Équations aux Dérivées Partielles*, Paris 1929, pp. 74–75, is particularly relevant. The other references are: C. Riquier, *Les Systèmes d'Équations aux Dérivées Partielles*, Paris, 1910, and J. M. Thomas, *Differential Systems*, New York, 1937.

[8] See the author's paper "The General Geometry of Paths," *Ann. Math.*, **29**, 143–168 (1928).

[9] For the definition of covariant differentiation, see formula (9.2) of the paper cited in the preceding footnote.

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# THE GENERALITY OF FINITE ABSTRACT COMPLEXES

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1. Let $A$ be a finite abstract complex as defined by S. Lefschetz[1] following A. W. Tucker, W. Mayer and J. W. Alexander. An *open* subcomplex $C$ of an abstract complex $D$ is a subset of $D$ (order, dimensions and incidences in $C$ determined by those in $D$) such that $x \in C$ and $y > x$ implies $y \in C$. It will be shown here that for every $A$ there is an open subcomplex $B$ of a simplicial complex such that the following homology groups using integer coefficients are isomorphic:

$$H^q(A) \approx H^{q+\alpha}(B), \qquad q \text{ arbitrary}, \tag{1}$$

where $\alpha = 0$ if $A$ has no elements of negative dimension and no zero-dimensional torsion coefficients and otherwise $\alpha > 0$. A result of Steenrod's[2] shows that relation (1) then holds for any coefficient group. One of the principal uses of an abstract complex being to carry a homology theory, the present result shows that in this respect and for the finite case simplicially realizable complexes are as general as any abstract complexes.

2. Let $\| b_{ij} \|$ be the normal form of the $p$, $p - 1$ incidence matrix of $A$, and give each row of this matrix a name, $E_i^p$, and each column a name $E_j^{p-1}$. If this is done for each row and column of all the simultaneously reduced incidence matrices of $A$, the set $\{E\}$ may be made an abstract complex $C$ by defining as incidence relations $[E_i^p : E_j^{p-1}] = b_{ij}$. Obviously

$$H^q(A) \approx H^q(C) \qquad q \text{ arbitrary}. \tag{2}$$

Because the normal matrices are diagonal and $FF = 0$ in $C$, $[E_i^p : E_j^{p-1}] \neq 0$ implies that all other incidence relations involving either $E_i^p$ or $E_j^{p-1}$ are zero.

3. Suppose that for some $i$ $[E_i^p : E_i^{p-1}] = k \neq 0, 1, -1$. This cannot happen in a simplicial complex, so to make $C$ simplicial each such pair