The discussion in Janet, *Leçons sur les Systèmes d'Équations aux Dérivées Partielles*, Paris 1929, pp. 74–75, is particularly relevant. The other references are: C. Riquier, *Les Systèmes d'Équations aux Dérivées Partielles*, Paris, 1910, and J. M. Thomas, *Differential Systems*, New York, 1937.

[8] See the author's paper "The General Geometry of Paths," *Ann. Math.*, **29**, 143–168 (1928).

[9] For the definition of covariant differentiation, see formula (9.2) of the paper cited in the preceding footnote.

THE GENERALITY OF FINITE ABSTRACT COMPLEXES

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1. Let A be a finite abstract complex as defined by S. Lefschetz[1] following A. W. Tucker, W. Mayer and J. W. Alexander. An *open* subcomplex C of an abstract complex D is a subset of D (order, dimensions and incidences in C determined by those in D) such that $x \in C$ and $y > x$ implies $y \in C$. It will be shown here that for every A there is an open subcomplex B of a simplicial complex such that the following homology groups using integer coefficients are isomorphic:

$$H^q(A) \approx H^{q+\alpha}(B), \qquad q \text{ arbitrary}, \tag{1}$$

where $\alpha = 0$ if A has no elements of negative dimension and no zero-dimensional torsion coefficients and otherwise $\alpha > 0$. A result of Steenrod's[2] shows that relation (1) then holds for any coefficient group. One of the principal uses of an abstract complex being to carry a homology theory, the present result shows that in this respect and for the finite case simplicially realizable complexes are as general as any abstract complexes.

2. Let $\| b_{ij} \|$ be the normal form of the p, $p - 1$ incidence matrix of A, and give each row of this matrix a name, E_i^p, and each column a name E_j^{p-1}. If this is done for each row and column of all the simultaneously reduced incidence matrices of A, the set $\{E\}$ may be made an abstract complex C by defining as incidence relations $[E_i^p : E_j^{p-1}] = b_{ij}$. Obviously

$$H^q(A) \approx H^q(C) \qquad q \text{ arbitrary}. \tag{2}$$

Because the normal matrices are diagonal and $FF = 0$ in C, $[E_i^p : E_j^{p-1}] \neq 0$ implies that all other incidence relations involving either E_i^p or E_j^{p-} are zero.

3. Suppose that for some i $[E_i^p : E_i^{p-1}] = k \neq 0, 1, -1$. This cannot happen in a simplicial complex, so to make C simplicial each such pair

E_i^p, E_i^{p-1} must be replaced by a complex D_i^p with the same homology groups and no incidence relations of absolute value greater than 1. The situation described in the last sentence of No. 2 makes it possible to do this without disturbing the rest of C.

I. The following adaptation to abstract complexes of "Subdivision by Section"[3] replaces a complex G by a complex G' *with the same homology groups:* If $E^p \in G$ and $FE^p = C_1^{p-1} + C_2^{p-1}$ where C_i^{p-1} are $(p-1)$-chains of G, E^p is replaced by e_1^p, e_2^p, e^{p-1} to form G', incidences being given by

$$\begin{aligned} Fe^{p-1} &= FC_2^{p-1} \\ Fe_1^p &= C_1^{p-1} + e^{p-1} \\ Fe_2^p &= C_2^{p-1} - e^{p-1} \\ [E^{p+1}:e_i^p] &= [E^{p+1}:E^p],\ i = 1, 2 \end{aligned}$$

and all other incidences (not involving E^p) are the same in G' as in G.

II. Without loss of generality k may be assumed > 1 and the subscripts i may be omitted from the E's. Using I replace E^p by g_1^p, e_1^p, e_1^{p-1} so that $Fe_1^{p-1} = 0$, $Fe_1^p = E^{p-1} + e_1^{p-1}$, $Fg_1^p = (k-1)E^{p-1} - e_1^{p-1}$, $[E^{p+1}:g_1^p] = [E^{p+1}:e_1^p] = [E^{p+1}:E^p] = 0$. Then unless $k - 1 = 1$ replace g_1^p by g_2^p, e_2^p, e_2^{p-1}, etc., until after $n = k - 1$ steps

$$\begin{aligned} Fe_i^{p-1} &= 0 \qquad 1 \leqslant i \leqslant k \\ Fe_i^p &= E^{p-1} - e_{i-1}^{p-1} + e_i^{p-1} \end{aligned} \tag{3}$$

where $e_0^{p-1} = e_k^{p-1} = 0$ and $e_k^p = g_n^p$, all other incidences involving these elements being zero. The situation is now as follows: k new elements e_i^p and $k - 1$ new elements e_i^{p-1} have replaced e^p. Examination of formula 3 shows that the incidence $[E^p:E^{p-1}] = k > 1$ has been replaced by several incidences all of absolute value 1; e_i^p and e_{i+1}^p have the two common faces E^{p-1} and e_i^{p-1}; e_i^{p-1} is oriented to e_i^p oppositely than to e_{i+1}^p, the last two of which circumstances are incompatible with simpliciality and impose the following changes:

III. Subdivide each e_i^p twice more by means of I, finally getting the set $\{s_i^p, s_i^{p-1}, t_i^p, t_i^{p-1}, u_i^p, e_i^{p-1}\} = D^p$, $i = 1, 2, \ldots, k$ with incidences given by

$$Fe_i^{p-1} = Ft_i^{p-1} = Fs_i^{p-1} = 0 \tag{4}$$

$$Fu_i^p = -e_{i-1}^{p-1} + e_i^{p-1} + t_i^{p-1} \tag{5}$$

$$Ft_i^p = -t_i^{p-1} - s_i^{p-1} \tag{6}$$

$$Fs_i^p = E^{p-1} + s_i^{p-1}, \tag{7}$$

all other incidences involving elements of D^p being zero.

IV. Clearly D^p is incidence-equivalent to an open 2-subcomplex K^2 of a simplicial complex where s_i^p, t_i^p, u_i^p are represented by 2-simplexes and e_i^{p-1},

s_i^{p-1}, t_i^{p-1} by suitably chosen 1-simplexes on the boundaries of the 2-simplexes. K^2 has no vertices and some of the 1-simplexes on boundaries of 2-simplexes are missing so it is simplicially open.

V. In order to preserve dimension as far as possible, a technique for raising the dimension of K^2 is needed. If $\alpha \geqslant 0$ is an integer let $L^{p+\alpha}$ be the join $K^2 \smile \sigma$ where σ is a new open $(p + \alpha - 3)$-simplex ($\sigma^{-1} = 1$). If now s_i^p is replaced by $s_i^p \smile \sigma$ and similarly for all simplexes of K^2 the incidence formulas 4–7 remain unchanged, so $H^{q+\alpha}(L^{p+\alpha})$ is isomorphic to the qth homology group of the complex $\{E^p, E^{p-1}\}$.

4. The final complex B is obtained as follows:

a. If the minimum dimension, m, of any element of A is $\geqslant 0$ and A has no zero-dimensional torsion coefficient, let $\alpha = 0$. If $m < 0$ and A has no m-dimensional torsion coefficient, let $\alpha = -m$. If $m \leqslant 0$ and A has an m-dimensional torsion coefficient let $\alpha = - m + 1$.

b. To each D_i^p (corresponding to a pair E_i^p, E_i^{p-1} such as considered in No. 3) assign its $L_i^{p+\alpha}$. To each E_i^p of C with $FE_i^p = 0$ assign an open $(p + \alpha)$-simplex $W_i^{p+\alpha}$ such that neither W nor its boundary FW meets any previously assigned simplex. The set $B = \{L_i^{p+\alpha}, W_i^{p+\alpha}\}$ is an open complex of a simplicial complex and

$$H^{q+\alpha}(B) \approx H^q(A). \qquad (1)$$

To see the latter notice that B has the same structure (except for dimension if $\alpha > 0$) as C with regard to all elements E_i^p of C which do not have the property $FE_i^p = \pm E_i^{p-1}$. These E_i^p are omitted from representation in B because they have no effect on the homology groups of C. Hence by formula 2,

$$H^{q+\alpha}(B) \approx H^q(C) \approx H^q(A)$$

which gives formula 1.

[1] Lefschetz, S., *Bull. Am. Math. Soc.*, **43**, 345–359 (1937). See there for references to other authors.

[2] Steenrod, N. E., *Amer. Jour. Math.*, **58**, 675 (1936).

[3] Lefschetz, S., "Topology," *Am. Math. Soc. Colloquium Publications*, **12**, 68 (1930). Replace proof there by its group-theoretic analogue.