ON THE EUCLIDEAN CONNECTIONS IN A FINSLER SPACE

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The generalization of the parallelism of Levi-Civita in a Riemann space to a Finsler space has been regarded as one of the most important problems of Finslerian Geometry. For its solution different suggestions were made by J. L. Synge, J. H. Taylor, L. Berwald and E. Cartan. In this note we shall study the problem by employing a different method—the method of equivalence. We shall prove that in a general Finsler space an infinite number of Euclidean connections can be defined, in which the connections defined by other authors are included as particular cases.

Let $x^i$ be the coordinates of an $n$-dimensional Finsler space, whose fundamental integral is

$$ s = \int_{\alpha}^{\beta} F\left(x^i, \frac{dx^i}{dt}\right) dt, \quad (1) $$

where $F$ is positively homogeneous of the first order in the last $n$ arguments:

$$ F(x^i, \lambda y^i) = \lambda F(x^i, y^i), \quad y^i = \frac{dx^i}{dt}, \quad \lambda > 0. \quad (2) $$

It is well known that the Pfaffian form

$$ \omega = \frac{\partial F}{\partial y^i} dx^i \quad (3) $$

is invariant (under a general point transformation). We adjoin to the coordinates $x^i$ of the space $n(n-1)$ auxiliary variables $y^s$, subjected to the conditions

$$ y^s y^k = 0, \quad (4) $$

and put

$$ y^s = \frac{\partial F}{\partial y^s}. \quad (5) $$

Then in the space of all the variables $x^i, y^i, y^s$ we have $n$ linearly independent invariant Pfaffian forms, namely,

$$ \omega^i = y^i dx^k, \quad (6) $$

where $\omega^n = \omega$. It is from the Pfaffian forms $\omega^i$ that we shall develop our invariant theory of Finsler spaces, from which the Euclidean connections in the space are derived as consequences.
We introduce the elements $u^k_i$ of the inverse matrix of $(v^i_k)$, so that we have

$$u^k_i v^i_k = v^i_i u^i_k = \delta^k_i.$$  \hspace{1cm} (7)

The elements $u^k_i$ do not depend on the auxiliary variables, since a comparison of equations (4), (5), (7) gives

$$u^k_i = \frac{1}{F} y^i.$$ \hspace{1cm} (8)

If we form the exterior derivative of $\omega^n$, we see that we can write

$$(\omega^n)' = [\omega^i \omega_n^i],$$ \hspace{1cm} (9)

where

$$\omega^n = -u^k_i \frac{\partial^2 F}{\partial y^j \partial y^k} dy^j + \frac{1}{F} u^i_a \left( \frac{\partial F}{\partial x^i} - \frac{\partial^2 F}{\partial x^i \partial y^k} y^k \right) \omega^n +$$

$$u^i_a u^k_\beta \frac{\partial^2 F}{\partial x^i \partial y^k} \omega^\beta + \lambda_{\alpha \beta} \omega^\alpha, \quad \lambda_{\alpha \beta} = \lambda_{\beta \alpha},$$ \hspace{1cm} (10)

the $n(n-1)/2$ quantities $\lambda_{\alpha \beta}$ being arbitrary.

We shall suppose that the fundamental integral (1) leads to a "regular problem" of the calculus of variations. As is well known, this amounts to assuming that the matrix

$$\left( \frac{\partial^2 F}{\partial y^i \partial y^j} \right)$$

is of rank $n-1$, or, in our notation, that the Pfaffian forms $\omega^i, \omega^n$ are linearly independent.

If we form the exterior derivative of $\omega^n$, we see that the following invariant conditions can be imposed:

$$(\omega^n)' = \delta^{\alpha \beta} [\omega^i \omega^n_i], \text{ mod. } \omega^n,$$ \hspace{1cm} (11)

where $\delta^{\alpha \beta}$ is Kronecker's symbol. The conditions (11) are equivalent to

$$u^i_a u^k_\beta \left( F \frac{\partial^2 F}{\partial y^i \partial y^k} \right) = \delta_{\alpha \beta},$$ \hspace{1cm} (12)

under which the variables $u^i_a$ are not all independent. By carrying out the calculation of $(\omega^n)'$, we get

$$(\omega^n)' - \delta^{\alpha \beta} [\omega^i \omega^n_i] = [\omega^i \omega^i_n],$$ \hspace{1cm} (13)

where

$$\omega^i_n = v^i_k u^k_\beta - \delta^\alpha \gamma \left( u^i_a u^k_\beta \frac{\partial^2 F}{\partial x^i \partial y^k} + \lambda_{\beta \gamma} \right) \omega^n + \mu^\alpha_\gamma \omega^\gamma, \hspace{1cm} (14)$$
the quantities $\mu_\gamma^\alpha$, being symmetric in $\beta$, $\gamma$ and being introduced to get the most general expression for $\omega^\alpha_\gamma$.

Since the variables $u^i$ are connected by the relations (12), the Pfaffian forms $\omega^\alpha_\gamma$ are not linearly independent. In fact, we find that they satisfy the relations

$$\omega_{\alpha\beta} + \omega_{\alpha\alpha} \equiv 0, \text{ mod. } \omega^i, \omega^\gamma_\gamma,$$

where the quantities $\delta_{\alpha\beta}$ are used to raise and lower indices, thus

$$\omega_{\alpha\beta} = \delta_{\beta\gamma} \omega^\gamma_\alpha, \quad \omega^\beta_\alpha = \delta^\gamma_\alpha \omega^\beta_\gamma, \quad \text{etc.} \quad (15)$$

Our fundamental result is that we can, by a proper choice of $\lambda_{\alpha\beta}$, $\mu_\gamma^\alpha$, arrive at the conditions

$$\omega_{\alpha\beta} + \omega_{\alpha\alpha} = H_{\alpha\beta} \omega^\gamma_\gamma, \quad (16)$$

and that under these conditions the quantities $\lambda_{\alpha\beta}$, $\mu_\gamma^\alpha$, and hence the Pfaffian forms $\omega^\alpha_\gamma$, $\omega^\gamma_\gamma$ are completely determined. These Pfaffian forms are therefore invariant Pfaffian forms. The quantities $H_{\alpha\beta\gamma}$ constitute the first set of invariants of the Finsler space. Their vanishing signifies that the Finsler space is a Riemann space.

The expressions for the exterior derivatives of $\omega^\alpha_\gamma$, $\omega^\gamma_\gamma$ will lead to further invariants of the space. To find them we write the equations (9), (13), (16) in the condensed form

$$[\omega^i, \omega^j, \omega^k] = \{ \omega^i, \omega^j, \omega^k \} \quad (17)$$

with the understanding that $H_{ijk}$ is zero when any one of its indices is $n$. Putting

$$\Omega^i_k = (\omega_k^i)' - [\omega_k^i, \omega_j^i], \quad (18)$$

we find, by simply applying the "theorem of Poincaré" that the exterior derivative of the exterior derivative of a Pfaffian form is zero, that $\Omega_k^i$ are of the form

$$\Omega_k^i = R_k^i,_{j1} \omega^j + P_k^i,_{j}^g [\omega_\alpha^\alpha, \omega^j], \quad (19)$$

where

$$R_k^i,_{j1} + R_k^i,_{j1} = 0 \quad (20)$$

For a function $F$ in our variables its differential $dF$ is a linear combination of $\omega^i$, $\omega_j^i$, the coefficients of the linear combination being the "covariant derivatives." The invariants $H_{ijk}$, $R_k^i,_{j1}$, $P_k^i,_{j}$ and their covariant derivatives form a complete system of invariants in the sense that they are sufficient to determine a Finsler space up to a point transformation.

Now we shall enter into the geometrical interpretation of our results. For this purpose we put
\[ \pi_{ij} = \omega_{ij} + \gamma_{i\alpha} \omega^{\alpha} \]

and impose the conditions

\[ \gamma_{i\alpha} + \gamma_{j\alpha} = -H_{ij\alpha} \]

in order to have

\[ \pi_{ij} + \pi_{ji} = 0. \]

We also put

\[ \pi^i = \omega^i. \]

The Pfaffian forms \( \pi^i, \pi_j^i \), of which the latter have not yet been completely determined, will then be employed to define the Euclidean connection in the space.

To each set of variables \( x^i, \gamma^i, v_\alpha^i \) we attach a Euclidean space of \( n \) dimensions with the frame of reference \( \overrightarrow{Me_1} \ldots \overrightarrow{Me_n} \), where \( M \) is a point and \( e_1, \ldots, e_n \) are \( n \) mutually perpendicular unit vectors through \( M \). The equations

\[ dM = \pi^i e_i \]
\[ de_i = \pi^i_i e_i \]

then determine the infinitesimal displacement between two neighboring Euclidean spaces or a Euclidean connection. The property of the Euclidean connection depends on the expressions for the following exterior quadratic differential forms

\[ \Pi^i = (\pi^i)' - [\pi^j \pi^i_j]. \]
\[ \Pi_j^i = (\pi_j^i)' - [\pi^j \pi^i_k]. \]

And we find

\[ \Pi^i = \Omega^i - \gamma_i^\alpha [\omega^j \omega^{\alpha j}], \]
\[ \Pi_j^i = \Omega_j^i + \gamma_j^\alpha [\omega^\alpha_i - \gamma_j^\alpha \gamma^\beta \omega^\beta_i + (d\gamma_j^i \gamma^i_\alpha + \gamma_j^\alpha \omega^{\alpha j} - \gamma_j^\beta \omega^\beta_i)] . \]

Due to a reason which we shall give later, it is important to impose the condition that \( \Pi^i, \Pi_j^i \) be exterior quadratic differential forms in \( \omega^i, \omega^{\alpha n} \).

This gives

\[ d\gamma_j^i \gamma^i_\alpha + \gamma_j^\beta \omega^\alpha_i - \gamma_j^\alpha \omega^\beta_i + \gamma_j^\beta \omega^\beta_i = 0, \text{ mod. } \omega^\beta, \omega^{\alpha n}. \]  

Summing up, the conditions on \( \gamma_{i\alpha} \) are (22), (28). We shall satisfy (22) by supposing

\[ \gamma_{i\alpha} = \gamma_{\alpha i\alpha} = 0, \]
\[ \gamma_{\alpha \beta} + \gamma_{\beta \alpha} = -H_{\alpha \beta}. \]
Then (28) is reduced to the following more symmetrical form:
\[ d\gamma_{\rho\sigma\alpha} + \gamma_{\rho\sigma\beta} \omega^\beta_{\alpha} + \gamma_{\rho\sigma\alpha} \omega^\beta_{\rho} + \gamma_{\rho\sigma\alpha} \omega^\rho_{\sigma} \equiv 0, \text{ mod. } \omega^i, \omega^i.* \] (30)

It is important to note that \( \gamma_{\rho\sigma\alpha} \) have naturally to be invariants. An example of such a set of invariants is furnished by \( H_{\rho\sigma\alpha} \). We can easily verify that the condition (30) for \( \gamma_{\rho\sigma\alpha} \) is equivalent to saying that \( \gamma_{\rho\sigma\alpha} \) is of the form
\[ \gamma_{\rho\sigma\alpha} = G^{ijk} \omega^i_\mu \omega^j_\nu \omega^k_\alpha, \] (31)
where \( G^{ijk} \) are functions of \( x^m, y^m \) only.

To each set of invariants \( \gamma_{\rho\sigma\alpha} \) satisfying the conditions (29), (30) we have the set of Pfaffian forms \( \pi^i, \pi_i^j \). With these Pfaffian forms the equations (25) define a Euclidean connection, which gives the infinitesimal displacement between the Euclidean spaces attached to two neighboring sets of variables \( x^i, y^i, v^a_i \). Owing to the property that \( \Pi^i, \Pi_i^j \) are exterior quadratic differential forms in \( \omega^i, \omega^i_n \), it follows that \( \Pi^i, \Pi_i^j \) are zero, when \( x^i, y^i \) are given as functions of a parameter \( t \):
\[ x^i = x^i(t), \quad y^i = y^i(t). \] (32)

Hence along a one-parameter family of contact elements \( (x^i, y^i) \) the system of equations (25) is completely integrable and the Euclidean spaces attached can be developed in one and the same Euclidean space. This is essentially the generalization of the well-known theorem of Fermi to a Finsler space. From this fact we see that we can regard the Finsler space as formed by the \( (2n - 1) \)-parameter family of its contact elements and define a Euclidean connection in the space, with a “tangent Euclidean space” attached to each contact element. There are as many Euclidean connections in the space as there are invariants satisfying our conditions. If we take
\[ \gamma_{\rho\sigma\alpha} = -1/2 H_{\rho\sigma\alpha}, \] (33)
we get the Euclidean connection defined by Cartan.

In conclusion, we remark that each of our Euclidean connections has the property that the equivalence of the Euclidean connection is a necessary and sufficient condition for the equivalence of the Finsler spaces (under point transformations).

5 It is agreed, throughout this paper, that a Latin index runs from 1 to \( n \) and a Greek index from 1 to \( n - 1 \).