The next four are typical of a class in which the sum of two terms can be written $x(y^n + w^n)$:

$$x(y^3 + w^3) + zyw, \quad [1, 3, 4, 6];$$
$$x(y^3 + w^3) + zy^2w^2, \quad [1, 3, 4, 5];$$
$$x(y^5 + w^5) + zyw, \quad [1, 5, 6, 10, 15, 20];$$
$$x(y^5 + w^5) + zy^4w^4, \quad [1, 2, 6, 7, 8, 9, 10]. \qquad (9)$$

As examples of forms with index different from [1],

$$xy^3 - yz^3 + zx^3, \quad [2];$$
$$x^4y + y^2xz^2 + z^5, \quad [3, 5, 12, 17];$$
$$x^5 + zx^2y^2 + y^5, \quad [2, 3, 5];$$
$$x^4y^2 + yz^5 + z^3x^3, \quad [5, 6, 11, 13]. \qquad (10)$$

5. A sufficient indication of a method for obtaining indices of forms of odd extent of any degree in any number of variables is given by two simple remarks: $I^{-1}[1]$ is the set of all integers; a solution of any one of a given set of equations $E_1 = 0, \ldots, E_s = 0$ is a solution of $E_1 \ldots E_s = 0$. (All of the examples in §4 refer to the case $s = 2$.) When at least one of the equations, say, $E_j = 0$, can be written $E_j' = E_j''$, where E_j', E_j'' are monomials in the variables and have no variable in common, the complete integer solution of $E_j = 0$ expresses each of the variables as a monomial in integer parameters. Applying this and the known methods for finding the solutions described to the second remark above, we find the indices. In all of the examples in §4 the coefficients in the forms are ± 1. Any numerical coefficients may be introduced as in extending the solution of $E_j' = E_j''$ to that of $a_jE_j' = b_jE_j''$, where a_j, b_j are any given integers. If each of $E_1 = 0, \ldots, E_s = 1$, $s > 1$ can be written in the same form as E_j, the extent of the form whose index is to be found is $2^s - 1$.

THE APOLLONIAN PACKING OF CIRCLES

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A theorem of the senior author concerning the covering of the plane by circles is established in *Scripta Mathematica* for March, 1943.[1] In this paper we give an analytic proof of the same theorem. The subject is also of interest in geology (packing of sand, porosity problems) and other fields of science.

1. *Definitions and Statement of the Theorem.* Let us effect a covering

of the plane with mutually external, equal circles each tangent to six others. We call this configuration the *circlex*. That part of the plane which is external to the circles of our covering consists of curvilinear triangles. In each of these we inscribe a circle, and continually keep inscribing circles in each curvilinear triangle obtained thereafter *ad infinitum*. We call the latter configuration an *Apollonian packing of circles in the plane* (*or hypercirclex*).

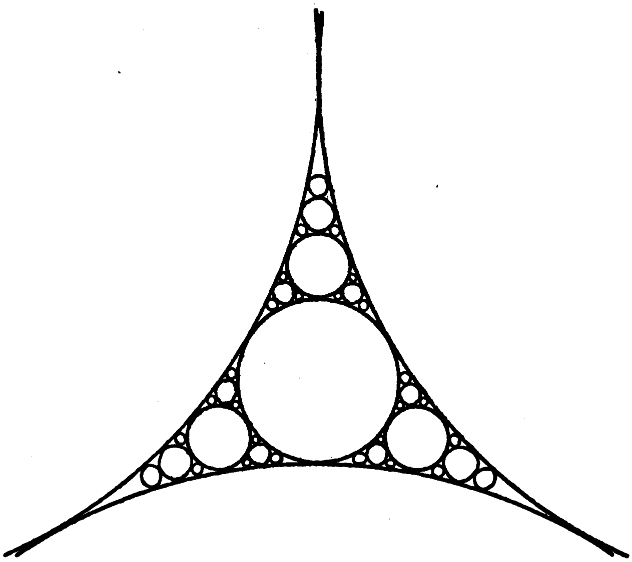

FIGURE 1

Let $T(a, b, c)$ denote the curvilinear triangle formed by the three mutually and externally tangent circles C_a, C_b, C_c with radii a, b, c, respectively. The term *Apollonian packing of circles in* $T(a, b, c)$ is defined similarly, i.e., we inscribe a circle in $T(a, b, c)$ and continually keep inscribing circles in each resulting curvilinear triangle (see Fig. 1).

Let a set G of mutually external circles each lying in $T(a, b, c)$ be given. Then the term *vacancy of* $T(a, b, c)$ *relative to* G signifies the set of points in or on $T(a, b, c)$ which are not interior to an element of G.

We now state the

THEOREM: *The area of the vacancy of an Apollonian packing of circles in any curvilinear triangle is zero. In other words, the sum of the areas of the infinity of circles of an Apollonian packing in* $T(a, b, c)$ *is equal to the area of* $T(a, b, c)$.

We note that since our theorem is true, then because of the symmetrical properties of our initial covering of the plane, it is an immediate consequence that the area of the vacancy of an Apollonian packing of circles in the plane is zero.

Our method of proof is different from that of the previous paper in the sense that we choose a different sequence of diminishing vacancies, it being easier to determine a bound for the ratio of diminution at each stage.

Let H denote the set of circles associated with an Apollonian packing in $T(a, b, c)$.

Let $C(x, y, z)$ denote the circle inscribed in $T(x, y, z)$.

The most natural method of exhausting the circles of H would be to group them into subdivisions as follows: We inscribe a circle in $T(a, b, c)$; then we inscribe a circle in each of the three curvilinear triangles formed; then in the 3^2 triangles formed, etc. Thus, for $n = 1, 2, 3, \ldots, i, \ldots$

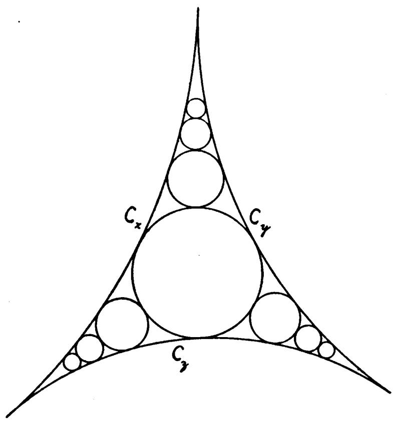

FIGURE 2

1st subdivision: $C(a, b, c)$

2nd subdivision: $C(C(a, b, c), b, c)$, $C(C(a, b, c), a, c)$, $C(C(a, b, c), a, b)$. If the circles of the $(n - 1)$st subdivision are:

$$C(a_1, b_1, c_1),\ C(a_2, b_2, c_2),\ \ldots,\ C(a_k, b_k, c_k)$$

where $k = 3^n - 2$, then the circles of the nth subdivision are:

$$C(C(a_j, b_j, c_j), b_j, c_j),\ C(C(a_j, b_j, c_j), a_j, c_j),\ C(C(a_j, b_j, c_j), a_j, b_j)$$

where $j = 1, 2, \ldots, 3^{n-2}$. But, the most natural procedure is not the most convenient in this case.

We define what we shall call a *necklace packing* (see Fig. 2) of a curvilinear triangle $T(a, b, c)$. It is the set of circles which comprises $C(a, b, c)$, the three circles inscribed in the curvilinear vertex triangles, the three circles inscribed in the new vertex triangles, and so on. Thus for $n = 1, 2, \ldots, i, \ldots$ it contains the circles belonging to all of the following subdivisions:

1st subdivision: $C(a, b, c)$

2nd subdivision: $C(C(a, b, c), b, c)$, $C(C(a, b, c), a, c)$, $C(C(a, b, c), a, b)$.

If the circles of the $(n - 1)$st subdivision are:

$$C(a_1, b, c),\ C(a_2, a, c),\ C(a_3, a, b)$$

then the circles of the nth subdivision are

$$C(C(a_1, b, c), b, c),\ C(C(a_2, a, c), a, c),\ C(C(a_3, a, b), a, b).$$

2. *The Fundamental Lemma.* We now establish the following

LEMMA: *A necklace packing of a curvilinear triangle $T(x, y, z)$ covers more than one-half the area of $T(x, y, z)$.*[2]

Let us effect a necklace packing of $T(x, y, z)$. Let U and V denote any two tangent circles of the necklace set, with radii u, v and centers s_u, s_v respectively. We connect s_u to s_v with a line segment. We also draw line segments from the points s_u, s_v to the points x_u, x_v at which U and V touch one of the sides of $T(x, y, z)$, say C_x (see Fig. 3). We note that

$$\measuredangle\, x_u s_u s_v + \measuredangle x_v s_v s_u \leq \pi. \quad (1)$$

Let $K[R]$ denote the area of the region R. We show that $K[T(u, v, x)]$ *is less than the sum of the areas of the sectors S_u, S_v of U and V included in the triangle* (s_u, s_v, s_x), where s_x is the center of C_x. Let $LT(u, v, x)$ denote the linear triangle joining the vertices of the curvilinear triangle $T(u, v, x)$. Then

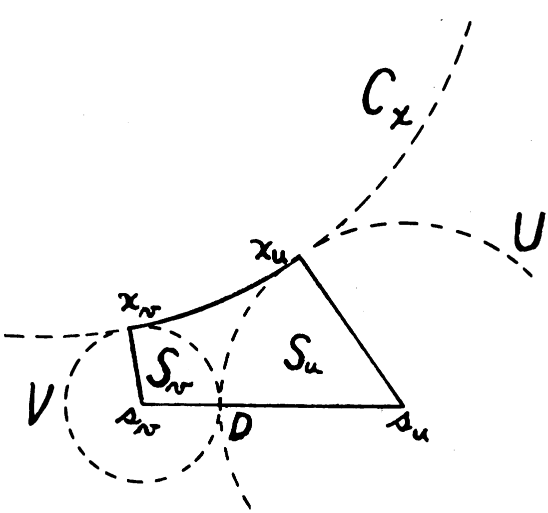

FIGURE 3

$$K[T(u, v, x)] < K[LT(u, v, x)]. \quad (2)$$

Now $LT(u, v, x)$ divides triangle (s_u, s_v, s_x) into four mutually exclusive triangles. We show that $K[LT(u, v, x)]$ *is less than the sum of the areas of those two of the other three triangles which contain the points s_u and s_v respectively.*

Let us consider the set of triangles with vertices R, S and T, where the points R and S are fixed, the segment $\overline{RS}$ being of constant length k, and where T may be any point in one of the half planes determined by the line going through R and S (see Fig. 4). We also admit the case where $\measuredangle RTS$ is zero, or $RT||TS$. Let P be any point on $\overline{RS}$ not an end-point. Let the segment $\overline{RP} = \alpha$ and the segment $\overline{PS} = \beta$. We lay off a segment $\overline{RQ}$ equal to α or $\overline{RT}$ and a segment $\overline{SW}$ equal to β on $\overline{ST}$. We show that *the area δ of Δ PQW is less than or equal to the sum μ of the areas of Δ RPQ and Δ PWS.*[3] Now

$$\delta = 2\alpha\beta \sin (R/2) \sin (S/2) \sin ((R + S)/2) \tag{3}$$

and

$$\mu = (\alpha^2/2) \sin R + (\beta^2/2) \sin S. \tag{4}$$

We must show that $\delta \leq \mu$ for

$$\sphericalangle R + \sphericalangle S \leq \pi. \tag{5}$$

From (5) we have

$$\sin (S/2) \leq \sin((\pi - R)/2) = \cos (R/2) \tag{6}$$

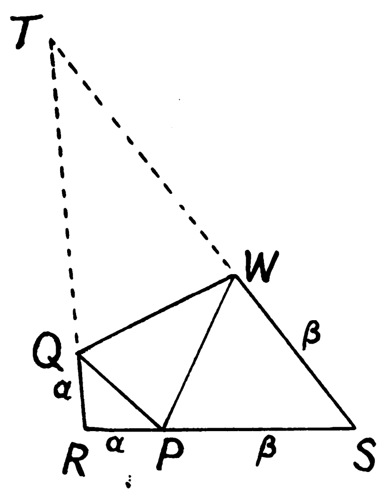

FIGURE 4

and

$$\sin (R/2) \leq \sin ((\pi - S)/2) = \cos (S/2). \tag{7}$$

Let us first consider the case where

$$\sin R \leq \sin S. \tag{8}$$

Then from (6) and (8) we have

$$\begin{aligned} &2\alpha\beta \sin (R/2) \sin (S/2) \sin ((R + S)/2) \\ &\leq 2\alpha\beta \sin (R/2) \sin (S/2) \\ &\leq 2\alpha\beta \sin (R/2) \cos (R/2) \\ &= \alpha\beta \sin R \\ &\leq ((\alpha^2 + \beta^2)/2) \sin R \\ &= (\alpha^2/2) \sin R + (\beta^2/2) \sin R \\ &\leq (\alpha^2/2) \sin R + (\beta^2/2) \sin S. \end{aligned} \tag{9}$$

Now let us consider the case where

$$\sin S < \sin R. \tag{10}$$

This case may be established by a sequence of inequalities as in (9), if we now use (7) where we have used (6), and (10) where we have used (8).

Now let D be the point at which the circles U and V are tangent. Then as a consequence of the latter argument we have

$$K[T(u, v, x)] < K[LT(u, v, x)] \leq K[\Delta(s_u, D, x_u)] + K[\Delta(s_v, D, x_v)] < K[S_u] + K[S_v]. \tag{11}$$

Now let $T(x, y, z)$ be completely subdivided as above. Thus, we join the centers of any two tangent circles of the necklace set associated with $T(x, y, z)$ with linear segments. And, if W is any circle of this necklace set, then we join its center to the points where W contacts C_x, C_y or C_z, with linear segments. Thus, $T(x, y, z)$ is the sum of a denumerable set of non-overlapping regions $\{A_i\}$, each bounded by three linear segments and a subarc of C_x, C_y or C_z (see solid line in Fig. 3). Let S_{i1} and S_{i2} be the two circular sectors included in A_i, and t_i the remaining curvilinear triangle. To each A_i we may now apply (11). Then

$$\begin{aligned} K[T(x, y, z)] &= \sum_{i=1}^{\infty} K[A_i] \\ &= \sum_{i=2}^{\infty}(K[S_{i1}] + K[S_{i2}] + K[t_i]) \\ &= \sum_{i=1}^{\infty} (K[S_{i1}] + K[S_{i2}]) + \sum_{i=1}^{\infty} K[t_i] \\ &\geq 2\sum_{i=1}^{\infty} K[t_i]. \end{aligned} \quad (12)$$

Therefore,

$$(\sum_{i=1}^{\infty} K[t_i])/K[T(x, y, z)] \leq 1/2. \quad (13)$$

This proves our fundamental lemma.

3. *Proof of the Theorem.* Let us effect a necklace packing of our curvilinear triangle $T(a, b, c)$. Let $\omega = K[T(a, b, c)]$. Denote the sum of the areas of the circles of this necklace set by σ_1. Then the area of the vacancy v_1 is

$$\omega - \sigma_1 \leq \omega/2. \quad (14)$$

But v_1 consists of a denumerable set of curvilinear triangles. In each of these effect a necklace packing and let σ_2 denote the sum of the areas of all these necklace sets. Then the area of our second vacancy v_2 is

$$\omega - (\sigma_1 + \sigma_2) \leq \omega/2^2. \quad (15)$$

Proceeding similarly, the area of the nth vacancy v_n is

$$\omega - \sum_{i=1}^{n} \sigma_i \leq \omega/2^n \quad (16)$$

and

$$\lim_{n \to \infty} \omega - \sum_{i=1}^{n} \sigma_i = 0. \quad (17)$$

But $\sum_{i=1}^{\infty} \sigma_i$ is the sum of the areas of the Apollonian packing of $T(a, b, c)$. Thus the area of the vacancy of the Apollonian packing of $T(a, b, c)$ is zero; or, the sum of the areas of the circles of an Apollonian packing in $T(a, b, c)$ is equal to the area of $T(a, b, c)$. Or, if we borrow the geological term *porosity*, and let it designate the ratio of the area of the vacancy (relative to a set of mutually external circles in $T(a, b, c)$) to the area of $T(a, b, c)$, then we may assert that the porosity of an Apollonian packing of circles in $T(a, b, c)$ is zero.

The packing of spheres requires methods which are essentially different from that of the present two-dimensional discussion. This question will be the subject of another paper.

In the case of the best covering of the plane with equal circles (no matter how small), the covering ratio is 0.9069, and thus the porosity is 0.0931. The covering ratio of space with equal spheres (no matter how small) is 0.7404 and thus the porosity is 0.2596 (normal packing). This is well known in geological literature in connection with the packing of sand and the amount of oil contained in the oil sands. Our work deals with the packing of unequal spheres, and we find that the porosity may be made as small as we please.[4]

[1] Kasner, E., Comenetz, G., and Wilkes, J., "The Covering of the Plane by Circles," *Scripta Mathematica*, **9**, 19–25 (1943).

[2] We have good reason to believe that the ratio of the sum of the areas of the circles of a necklace packing in any $T(x, y, z)$, to the area of $T(x, y, z)$, is greater than $\pi/4 = 0.7854$ and less than 0.8223. This problem will be treated in another paper. We thank Aida Kalish and Hüseyin Demir for assistance in the calculation of the upper limit 0.8223.

[3] We note that this part of the proof is somewhat more general than that actually required by the lemma.

[4] Kasner, E., "Note on Non-Apollonian Packing in Space," *Scripta Mathematica.*, **9**, 26 (1943). See also *Science*, Oct. 16, 1942. Incidentally we note the following new theorem: In any circular triangle (bounded by mutually tangent circles) the conformal bisectors of the three horn angles (these are necessarily circles) are concurrent. The resulting point we call the *inversive center* of the triangle. For the bisection of analytic horn angles see the senior author's papers an conformal geometry.

CONCERNING CONTINUA WHICH HAVE DENDRATOMIC SUBSETS

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Communicated October 16, 1943

The subset K of the compact continuum M is said to be a *dendratomic* subset of M if there exists an upper semicontinuous collection G of mutually exclusive continua filling up M such that (1) K is an element of G, (2) G is a dendron with respect to its elements and (3) if H is an upper semicontinuous collection of mutually exclusive continua filling up M such