TORUS HOMOTOPY GROUPS*

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A promising line of attack on the unsolved problem of calculating the homotopy groups π_r, $r = 1, 2, \ldots$, of a topological space Y was initiated by J. H. C. Whitehead.[1] The results which he has obtained involve a certain "multiplication": for any pair of elements $\alpha \in \pi_m$, $\beta \in \pi_n$ the Whitehead product $\alpha \cdot \beta$ is defined and is an element of the group π_{m+n-1}. This multiplication can be compared with the group operation only when $m = n = 1$ and in this case $\alpha \cdot \beta = \alpha\beta\alpha^{-1}\beta^{-1}$.

In order to study this product I define here, for every dimension $r \geqslant 1$, a group τ_r which I call the r-dimensional torus homotopy group and which has the following properties:

1. Every homotopy group of dimension less than $r + 1$ can be mapped isomorphically into τ_r. (In fact most homotopy groups have many such isomorphisms.)

2. If $\gamma = \alpha \cdot \beta$, where $\alpha \in \pi_m$, $\beta \in \pi_n$, $\gamma \in \pi_{m+n-1}$ and $m + n - 1 < r + 1$, then isomorphisms $\pi_m \to \tau_r$, $\pi_n \to \tau_r$, $\pi_{m+n-1} \to \tau_r$ can be so chosen that $\alpha \to \bar{\alpha}$, $\beta \to \bar{\beta}$, $\gamma \to \bar{\gamma}$ and $\bar{\gamma} = \bar{\alpha}\bar{\beta}\bar{\alpha}^{-1}\bar{\beta}^{-1}$.

The groups $\tau_1, \tau_2, \tau_3, \ldots$ form a direct homomorphism system and the limit group τ has properties (1) and (2) with $r = \infty$. Thus all the homotopy groups and all the Whitehead products can be studied within one single group τ.

Let Y be a topological space and y_* one of its points. Denote by T_{r-1} the $(r - 1)$-dimensional torus whose coordinates are the $r - 1$ real numbers $x_1, x_2, \ldots, x_{r-1}$ (mod 1). The r-dimensional torus homotopy group $\tau_r(Y) = \tau_r(Y, y_*)$ is defined to be the fundamental group of the function space $Y^{T_{r-1}}$, using the mapping $y_* = y_*^{T_{r-1}}$ as base point. Thus an element of $\tau_r(Y)$ is represented by a continuous Y-valued function f of r real variables $x_0, x_1, \ldots, x_{r-1}$ (mod 1) which satisfies the condition

$$(*) \qquad f(x_0, x_1, \ldots, x_{r-1}) = y_* \text{ if } x_0 \equiv 0.$$

Denoting by $\mathfrak{L}_r$ the totality of such functions f, it is easy to see that two such functions f_0 and f_1 determine the same element of τ_r whenever there is a homotopy f_t, $0 \leq t \leq 1$, between them such that $f_t \in \mathfrak{L}_r$ for every t. The multiplication in τ_r is determined by the multiplication in $\mathfrak{L}_r$: $h = f \cdot g$ is defined by

$$h(x_0, x_1, \ldots, x_{r-1}) = \begin{cases} f(2x_0, x_1, \ldots, x_{r-1}) \text{ when } 0 \leq x_0 \leq 1/2, \\ g(2x_0 - 1, x_1, \ldots, x_{r-1}) \text{ when } 1/2 \leq x_0 \leq 1. \end{cases}$$

The discussion of the preceding paragraph parallels a discussion of the classical homotopy groups[2] which is presumedly known to the reader. It is sufficient here to recall that an element of $\pi_m(Y) = \pi_m(Y, y_*)$ is represented by a continuous Y-valued function f of m real variables $x_0, x_1, \ldots, x_{m-1}$ (mod 1) which satisfies the condition

$$(**) \qquad f(x_0, x_1, \ldots, x_{m-1}) = y_* \text{ if } x_i \equiv 0 \text{ for some } i = 0, 1, \ldots, m-1.$$

Let the totality of such functions f be denoted by $\mathfrak{F}_m$. If $m \leq r$ and A denotes a sequence $\{i_1, i_2, \ldots, i_{m-1}\}$, where $0 < i_1 < \ldots < i_{m-1} < r$, then, by assigning to every $f \in \mathfrak{F}_m$ that $g \in \mathfrak{L}_r$ which is defined by

$$g(x_0, x_1, \ldots, x_{r-1}) = f(x_0, x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}),$$

a homomorphism ω_r^A: $\alpha \rightarrow \alpha^A$ of $\pi_m(Y)$ into $\tau_r(Y)$ is set up. These homomorphisms can be shown to be isomorphisms, although the proof of this fact is by no means easy. There are exactly $\binom{r-1}{m-1}$ of these isomorphisms $\pi_m \rightarrow \tau_r$, and there is no reason for preferring any of them over the others. It is easily seen that τ_1 is identical with π_1, and that τ_2 is identical with Abe's[3] group κ_2.

A homomorphism Φ of τ_r onto τ_{r-1} is defined by assigning to every $f \in \mathfrak{L}_r$ that $g \in \mathfrak{L}_{r-1}$ which is defined by

$$g(x_0, x_1, \ldots, x_{r-2}) = f(x_0, x_1, \ldots, x_{r-2}, 0).$$

It can be shown that the subgroup τ_r' of τ_r which is generated by the subgroups $\omega_r^A(\pi_m)$, $2 \leq m \leq r'$, $A \subset \{1, 2, \ldots, r-1\}$, is the direct product $\mathrm{X}\omega_r^A(\pi_m)$ of these subgroups, that τ_r' is the nucleus of the homomorphism Φ and that τ_r is a split extension of the abelian subgroup τ_r' by τ_{r-1}.

In general the torus homotopy groups are non-abelian. In particular τ_r is non-abelian in the case that Y is the union of an m-sphere and an n-sphere $(m + n - 1 = r)$ which have exactly one point in common. Thus τ_r may be non-abelian for simply connected spaces if $r \geq 3$ and for spaces with abelian fundamental group if $r \geq 2$. This shows that not all of the "non-abelian character" of a space is expressed by its fundamental group. However if Y is a topological group its torus homotopy groups are all abelian.

The exact statement of how the Whitehead products are represented in τ_r as commutators is as follows: *If A has $m - 1$ indices and B has $n - 1$ indices, if $A \frown B$ is vacuous and if $\alpha \in \pi_m(Y)$ and $\beta \in \pi_n(Y)$ then $(\alpha \cdot \beta)^{A \smile B} = [\alpha^A, \beta^B]^{\epsilon\eta}$, where $[u, v]$ denotes the commutator $uvu^{-1}v^{-1}$; $\epsilon = (-1)^{n-1}$ and $\eta = (-1)^w$ where w is the number of instances of $i > j$ with $i \in A$ and $j \in B$.* It should be observed that it follows from previous discussion that $[\alpha^A, \beta^B] = 1$ if $A \frown B$ is not vacuous. By the theory of group-extensions it can be shown that τ_r is determined by the groups $\pi_1, \pi_2, \ldots, \pi_r$ and the Whitehead products $\alpha \cdot \beta$, $\alpha \in \pi_m$, $\beta \in \pi_n$, $m + n - 1 < r + 1$.

By restating in terms of torus homotopy groups the combination of two of J. H. C. Whitehead's results[1] we find the following theorem: *Let K be a complex obtained from a complex K^* by removing the interior $\sigma - \dot{\sigma}$ of a principal n-dimensional simplex σ, where $n > 2$. The nucleus of the injection homomorphism $\tau_n(K) \to \tau_n(K^*)$ is precisely the invariant subgroup of $\tau_n(K)$ which is generated by the image of the injection homomorphism $\tau_n(\dot{\sigma}) \to \tau_n(K)$.* This reformulation seems to have more intuitive content and suggests more clearly than the original theorems an attack on the harder problem where $\dim \sigma > n$.

If Y_* is a subset of Y which contains y_* the relative torus homotopy group $\tau_r(Y \bmod Y_*, y_*)$, $r \geq 2$, may be defined. One starts with the collection $\mathfrak{L}_r(Y \bmod Y_*, y_*)$, of those continuous Y-valued functions f of the $r - 1$ real variables $x_0, x_1, \ldots, x_{r-2}$ (mod 1) and the real variable $0 \leq t \leq 1$ which satisfy the conditions

$$(***) \qquad \begin{cases} f(x_0, x_1, \ldots, x_{r-2}, t) \in Y_* \text{ if } t = 0, \\ f(x_0, x_1, \ldots, x_{r-2}, t) = y_* \text{ if } t = 1 \text{ or if } x_0 \equiv 0. \end{cases}$$

From $\mathfrak{L}_r(Y \bmod Y_*, y_*)$ the group $\tau_r(Y \bmod Y_*, y_*)$ is obtained by the same procedure which led to the definition of $\tau_r(Y, y_*) = \tau_r(Y \bmod y_*, y_*)$. Just as in the absolute case there are isomorphisms ω_r^A: $\pi_m(Y \bmod Y_*) \to \tau_r(Y \bmod Y_*)$, where $A \subset \{1, 2, \ldots, r - 2\}$, and homomorphisms Φ: $\tau_r(Y \bmod Y_*) \to \tau_{r-1}(Y \bmod Y_*)$. The subgroup $\tau_r'(Y \bmod Y_*)$ of $\tau_r(Y \bmod Y_*)$ which is generated by the subgroups $\omega_r^A(\pi_m(Y \bmod Y_*))$, $m \geq 3$, $A \subset \{1, 2, \ldots, r - 2\}$, is the direct product of these subgroups, τ' is the nucleus of Φ and τ is a split extension of τ_r' by τ_{r-1}

A homomorphism of $\tau_{r+1}(Y \bmod Y_*)$ into $\tau_r(Y_*)$ is defined by the transformation $f \to f \mid \{t = 0\}$; a homomorphism of $\tau_r(Y_*)$ into $\tau_r(Y)$ is defined by injection of Y_* into Y; a homomorphism of $\tau_{r+1}'(Y)$ into $\tau_{r+1}(Y \bmod Y_*)$ is defined by $f \to g$ where $g(x_0; x_1, \ldots, x_{r-1}; x_r) = f(x_0, x_A)$, $A \subset \{1, 2, \ldots, r\}$, $2 \leq m \leq r + 1$. In the resulting system of homomorphisms

$$\tau_{r+1}'(Y) \to \tau_{r+1}(Y \bmod Y_*) \to \tau_r(Y_*) \to \tau_r(Y)$$

it may be verified that the nucleus of any homomorphism is the image of the preceding homomorphism and that the nucleus of the first homomorphism is generated by the images of the injection homomorphisms $\pi_m(Y_*) \to \pi_m(Y)$, $2 \leq m \leq r + 1$. If Y is a fibre space over a space Z and Y_* is the fibre which contains y_* and whose image point is the base point z_* of Z then $\tau_r(Y \bmod Y_*, y_*) \approx \tau_r(Z, z_*)$ and the homomorphism system becomes

$$\tau_{r+1}'(Y) \to \tau_{r+1}(Z) \to \tau_r(Y_*) \to \tau_r(Y).$$

Any element of τ_r determines, together with the fundamental r-cycle of the antecedent space, a continuous r-dimensional cycle. In this fashion a homomorphism $\tau_r(Y)$ into the r-dimensional homology group $H_r(Y)$ of Y is defined. The image group is the group of spherical r-dimensional cycles, and the nucleus of this homomorphism obviously contains the commutator subgroup of τ_r. However, it is not true that the commutator subgroup is the nucleus. This is shown by the example of the 3-sphere $Y = S^3$ with $r = 4$. It is known that there is a non-trivial element α of $\pi_4(S^3)$, hence α^A is an element of the nucleus; on the other hand $\tau_4(S^3)$ is abelian because S^3 is a group manifold.

Elements of the nucleus which are not commutators may be constructed with the help of the homotopy groups of the rotation groups. If $\alpha \in \pi_n(Y)$ and ρ is an element of $\pi_m(R_{n-1})$, where R_{n-1} is the rotation group of the $(n - 1)$-sphere then there is determined an element $\alpha^\rho \in \tau_{m+n}(Y)$. The element $(\alpha^\rho)^A(\alpha^{-1})^B$, where $A = \{1, 2, \ldots, m + n - 1\}$ and $B = \{1, 2, \ldots, n - 1\}$, belongs to the nucleus of the homomorphism $\tau_{m+n}(Y) \rightarrow H_{m+n}(Y)$. If α is the generator of $\pi_2(S^2)$ and ρ is the generator of $\pi_1(R_1)$ then $(\alpha^\rho)^A(\alpha^{-1})^B$ is the generator of $\pi_3(S^2)$. From the fact that the "Einhängung" of this element is non-trivial it can be shown that $(\alpha^\rho)^A(\alpha^{-1})^B$ is not a commutator.[4]

The torus homotopy groups have a rather obvious generalization which may turn out to be useful. Let μ be any mapping of T_{r-1} into Y and consider the fundamental group $\tau_r(Y, \mu)$ of the function space $Y^{T_{r-1}}$ using the mapping μ as base point. If $\mu = y_*$ the definition reduces to that of $\tau_r(Y, y_*)$. The group $\tau_r(Y, \mu)$ depends only on r, Y and the homotopy class of μ; particularly, to any element α of $\tau_r(Y, \mu)$ there is a group $\tau_{r+1}(Y, \alpha)$. The cycles which arise from the homomorphism of $\tau_r(Y, \mu)$ into $H_r(Y)$ need not be spherical cycles; for example, the fundamental cycle of the torus can occur.

A more detailed analysis of the torus homotopy groups will appear elsewhere.

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[1] *Ann. Math.*, **42**, 409–428 (1941) and *Proc. London Math. Soc.*, **45**, 243–327 (1939), §§10, 11.

[2] All homotopy groups are multiplicative in this paper.

[3] *Jap. Jour. Math.*, **16**, 169 (1940).

[4] In this and the succeeding paragraph there is a certain amount of overlapping with recent unpublished work by George W. Whitehead.