Let X, Y be topological spaces, f a map $X \rightarrow Y$, and let $u \in H^p(Y)$, $v \in H^q(Y)$ satisfy (1). Let C be the mapping cylinder of f, and $F : C \rightarrow Y$ its projection. Let

$$i: X \rightarrow C, \quad j: (C, 0) \rightarrow (C, X)$$

be identity maps. Since F is homotopic to the identity map of C, F^* is isomorphic. Let $\bar{u} = F^*u$, $\bar{v} = F^*v$. Then $\bar{u} \smile \bar{v} = F^*(u \smile v) = 0$. Furthermore $i^*\bar{u} = i^*F^*u = (Fi)^*u = f^*u = 0$. By exactness of the cohomology sequence of (C, X), there exists a $u' \in H^p(C, X)$ such that $j^*u' = \bar{u}$. Form $u' \smile \bar{v} \in H^{p+q}(C, X)$. Then $j^*(u' \smile \bar{v}) = j^*u' \smile \bar{v} = \bar{u} \smile \bar{v} = 0$. By exactness of the cohomology sequence of (C, X), there exists an element $w \in H^{p+q-1}(X)$ such that $\delta w = u' \smile \bar{v}$. An examination of the effect of choosing different elements u', w shows that $w = -[f, u, v]$ is unique mod$[f^*H^{p+q-1}(Y) + H^{p-1}(X) \smile f^*v]$.

[1] *Math. Annalen*, **104**, 637–665 (1931); *Fund. Math.*, **25**, 427–440 (1935).

[2] *Comment. Math. Helv.*, **14**, 61–122 (1942).

[3] *Bull. Amer. Math. Soc.*, **43**, 785–805 (1937).

[4] See also "An Expression of Hopf's Invariant as an Integral," *Proc. Nat. Acad. Sci.*, **33**, 117–123 (1947).

[5] "Products of Cocycles and Extensions of Mappings," to appear in the *Annals of Math.*, No. 2 (1947).

THE (L^2)-SPACE OF RELATIVE MEASURE

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1. Let a real- or complex-valued function $f = f(x)$, where $0 < x < \infty$, be called of class (N^2) if it satisfies the following conditions: $f(x)$ is of class (L^2) on every bounded interval $(0, X)$ and the mean-value, $M(|f|^2)$, of $|f|^2$ exists as a finite limit, where the operator M is defined by

$$M(g) = \lim_{X \rightarrow \infty} \int_0^X g(x)dx/X. \tag{1}$$

If f and g are of class (N^2), then $\bar{M}(|f - g|^2) < \infty$, where

$$\bar{M}(p) = \limsup_{X \rightarrow \infty} \int_0^X p(x)dx/X, \qquad (p \geqq 0). \tag{2}$$

Since $\bar{M}(p_1 + p_2) \leqq \bar{M}(p_1) + \bar{M}(p_2)$, it follows that a metric function space, to be called the (N^2)-space, can be defined as follows: The elements of the space consist of all functions of class (N^2), and two elements, f and g, of the space are considered as identical if their distance is zero, with the under-standing that the distance is defined as the non-negative square root of $\bar{M}(|f - g|^2)$.

It is clear that, when $f(x)$ is periodic (for $x > 0$), it is of class (N^2) if and only if it is of class (L^2) over a period, and that the metric of the space (N^2) then becomes identical with the metric of the corresponding (L^2)-subspace. More generally, it is clear that, if almost-periodicity (B^2) is referred to the half-line $x > 0$, then $f(x)$ is of class (N^2) whenever it is of class (B^2), and that the metric of the space (N^2) becomes identical with the metric of its (B^2)-subspace. However, while both subspaces (L^2), (B^2) are linear vector spaces, it turns out that the full space (N^2) is not linear. On the other hand, the completeness property of the subspaces (L^2), (B^2) (Fischer-Riesz, Besicovitch) proves to be shared by the full space (N^2).

2. Let $f(x)$ be called of class $(N^2)_0$ if it is a bounded function of x and is of class (N^2), and let the metric space $(N^2)_0$ be defined as the function space consisting of the functions of class $(N^2)_0$ and carrying the metric of the (N^2)-space.

By considerations rediscovered by Besicovitch (1926), Nalli[1] (1914) proved that, if a sequence of elements in $(N^2)_0$ is a Cauchy sequence, then it has a limit in (N^2). In addition, Nalli[2] emphasized that her result implies an analogue of the Fischer-Riesz theorem for the "orthogonal system" $e^{i\lambda x}$, $-\infty < \lambda < \infty$, of the (B^2)-space.

However, the $(N^2)_0$-space is not complete. In order to see this, it is sufficient to consider the sequence of partial sums of the Fourier series of any periodic function of class (L^2) which fails to be bounded (even if x-sets of measure zero are disregarded).

3. *The space (N^2) is not a linear space.*[3]

This is equivalent to the statement that f_1 and f_2 can be in (N^2) when $f_1 + f_2$ is not. When f_1 and f_2 are real, this, in turn, is equivalent to the statement that the mean-value, (1), of the product $g = f_1 f_2$ of two functions of class (N^2) need not exist. Hence, if f_1 is chosen to be the constant 1 and if f_2 is denoted by f, it follows that it is sufficient to ascertain the existence of a function f of class (N^2) for which $M(f)$ fails to exist. But such an f results by choosing $f(x)$ to be $+1$ or -1 according as x is or is not in S, where S is a sequence of x-intervals having the following property: The mean-value $M(h)$ does not exist if h is the characteristic function of S (that is, $h(x)$ is 1 or 0 according as x is or is not in S). Since the existence of such S or $h(x)$ is obvious, it follows that (N^2) is not linear.

There does not exist in (N^2) a linear subspace containing all linear subspaces of (N^2).

This contains the preceding fact (since otherwise the space (N^2) itself would be an extremal linear subspace), but the converse deduction is also possible. In order to see this, it is sufficient to observe that the constant multiples of any f contained in (N^2) form a linear subspace of (N^2).

4. *The space* (N^2) *is complete.*

In other words, if $f_1, f_2, \ldots$ is any sequence of functions contained in (N^2) and satisfying

$$\bar{M}(|f_j - f_k|^2) \to 0 \text{ as } j, k \to \infty, \tag{3}$$

then there exists in (N^2) a function f satisfying

$$\bar{M}(|f - f_j|^2) \to 0 \text{ as } j \to \infty. \tag{4}$$

The proof is a refinement, and at the same time a simplification, of the proofs of Besicovitch[4] and of Nalli.[5]

The limit relation (3) obviously implies the existence of an unbounded increasing sequence of positive numbers $R(1), R(2), \ldots$ such that

$$\lim_{n \to \infty} \sigma(n) = 0,$$

where

$$\sigma(n) = \operatorname*{fin\,sup}_{m > n,\, X > R(m)} \int_0^X |f_m(x) - f_n(x)|^2 dx/X, \tag{5}$$

if the fin sup refers to m, while n is fixed. Let $n_1 < n_2 < \ldots$ be a sequence of integers satisfying

$$\sum_{k=1}^{\infty} \sigma(n_k) < \infty. \tag{6}$$

Let g_k denote the n_k-th function of the sequence $f_1, f_2, \ldots$ and let $S(k) = R(n_k)$. Put $f(x) = 0$ if $0 < x < S(1)$ and $f(x) = g_k(x)$ if $S(k) \leqq x < S(k + 1)$, where $k = 1, 2, \ldots$.

It will first be shown that

$$\bar{M}(|f - g_k|^2) \to 0 \text{ as } k \to \infty. \tag{7}$$

To this end, let X be any number exceeding $S(k)$, where k is fixed. If $i\ (\geqq k)$ denotes the integer for which $S(i) < X \leqq S(i + 1)$, then

$$\int_{S(1)}^{X} |f(x) - g_k(x)|^2\, dx = \sum_{j=1}^{k-1} \int_{S(j)}^{S(j+1)} |g_j(x) - g_k(x)|^2 dx + \sum_{j=k+1}^{i-1} \int_{S(j)}^{S(j+1)} |g_j(x) - g_k(x)|^2 dx + \int_{S(i)}^{X} |g_i(x) - g_k(x)|^2 dx.$$

Hence it is seen from (6) and the definitions of g_j and $S(j)$, that

$$\int_{S(1)}^{X} |f(x) - g_k(x)|^2 dx/X \leqq \sum_{j=1}^{k-1} \int_{S(j-1)}^{S(j)} |g_j(x) - g_k(x)|^2 dx/X + \sum_{j=k+1}^{i} \sigma(n_j).$$

Consequently, as $X \to \infty$,

$$\bar{M}(|f - g_k|^2) \leqq \sum_{j=k+1}^{\infty}\sigma(n_j),$$

and so (7) follows from (6).

It is clear that (4) follows from (7), and from the definition of g_k, in virtue of the triangular inequality.

It remains to be shown that $f(x)$ is of class (N^2). Clearly, the existence of (1) for $g = |f_n|^2$, where $n = 1, 2, \ldots$, and (3) imply that

$$\mu = \lim_{n\to\infty} M(|f_n|^2) \qquad (8)$$

exists as a finite limit. On the other hand, by Minkowski's inequality,

$$\left|\int_0^X |f(x)|^2dx - \int_0^X |f_n(x)|^2dx\right|/X < \int_0^X |f(x) - f_n(x)|^2dx/X; \quad (9)$$

hence, if the integer n is fixed so that neither $|\mu - M(|f_n|^2)|$ nor $\bar{M}(|f - f_n|^2)$ exceeds a given $\epsilon > 0$, then, whenever X exceeds a bound depending on ϵ and n,

$$\left|\int_0^X |f(x)|^2\, dx/X - \mu\right| < 2\epsilon.$$

Consequently, (1) exists for $g = |f|^2$, and

$$M(|f|^2) = \mu. \qquad (10)$$

5. Let a function $f(x)$, where $0 < x < \infty$, be called of class $(\bar{N}^2)$ if $f(x)$ is of class (L^2) on every bounded interval $(0, X)$ and $\bar{M}(|f|^2)$ is finite. The functions of class $(\bar{N}^2)$ will be considered to form a metric space carrying the same metric as that of the (N^2)-space; so that the (N^2)-space is a subspace of the $(\bar{N}^2)$-space.

The $(\bar{N}^2)$-space is linear and complete.

The linearity of $(\bar{N}^2)$ follows from the inequality mentioned at the beginning (after the definition of $\bar{M}$). On the other hand, the completeness is a consequence of the above proof, even though not of the final wording, of section 4.

6. It is clear that the above considerations can be adapted to any of the spaces (N^p), where $p \geqq 1$, which correspond to the space (N^2) in the same way as the classes (L^p) relate to the class (L^2).

If $p \geqq 1$ is arbitrary, the space (N^p) is complete but not linear.

It is understood that, if $p = 1$, the class $(N) = (N^1)$ is meant to be defined as consisting of those functions f which are of class $(L) = (L^1)$ on every bounded x-interval and satisfy the condition that $M(|f|)$ exists as a finite limit.

If $p > 1$ and $p^{-1} + q^{-1} = 1$, then the product of two functions, one of which belongs to (B^p) and the other to (B^q), belongs to $(B) = (B^1)$. But this becomes false if (B^p), (B^q) and (B) are replaced by (N^p), (N^q) and (N), respectively. This follows, even for $p = 2 = q$, from the above example proving that (N^2) is not linear. Correspondingly, (N^q) *cannot be interpreted as the dual space of* (N^p), since such an interpretation would involve the definition of a scalar product.

7. Let $(\overline{N}^p)$ denote the space which relates to the space (N^p) in the same way as the space $(\overline{N}^2)$ relates to (N^2).

If $p \geqq 1$ *is arbitrary, the space* $(\overline{N}^p)$ *is complete and linear.*

The proofs are the same as before.

The situation can be summarized as follows: (B^p) is a subspace of (N^p) and (N^p) is a subspace of $(\overline{N}^p)$; all three of these spaces are complete; the first and third of them are linear but the second is not.

[1] Actually, the definitions of Nalli (*loc. cit.*,[2] p. 306) are based on what results when the distance is assigned under the assumption that the upper limit occurring in (2) is replaced by the corresponding limit (which is required to exist). However, a glance at the proofs given by Nalli shows that this definition of her function space is equivalent to the above definition of the function space $(N^2)_0$.

[2] Nalli, P., *Rendiconti Circolo Matematico Palermo*, **38**, 307, 318–319, 322–323 (1914).

[3] This fact has curious methodical consequences for a problem relating to the Riemann zeta-function; cf. Wintner, A., *Duke Mathematical Journal*, **10**, 430 (1943), where the situation is explained in detail.

[4] Besicovitch, A. S., *Almost Periodic Functions*, Cambridge, 110–112 (1932).

[5] Nalli, P., *loc. cit.*[2] pp. 308–309.

ON THE MAXIMUM PARTIAL SUM OF INDEPENDENT RANDOM VARIABLES

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Let X_n, $n = 1, 2, \ldots$ be independent random variables with moments

$$E(X_n) = 0, \qquad E(X_n^2) = 1, \qquad E(|X_n|^3) = \gamma_n. \tag{1}$$

The reduction of the variance to 1 is not necessary, but is made here solely for the sake of simplicity. Let

$$S_n = \sum_{v=1}^{n} X_v, \qquad S_n^* = \underset{1 \leq v \leq n}{\text{Max.}} |S_v|$$

It is a trivial observation that the same law of the iterated logarithm holds for S_n^* as for S_n. In particular one of Feller's theorems[1] becomes: If

$$\text{sup. } |X_n| = 0\,(n^{1/2}(lg_2 n)^{-3/2}), \tag{2}$$

then ("*i. o.*" standing for "infinitely often").