[1] Chapman, S., and Cowling, T. G., *The Mathematical Theory of Non-Uniform Gases*, Cambridge, 1939, see § 15.3 and § 15.4.

[2] Tsien, H. S., "Superaerodynamics, Mechanics of Rarefied Gases," *J. Aero. Sci.*, **13**, 653–664 (1946).

[3] Schamberg, R., *The Fundamental Differential Equations and the Boundary Conditions for High Speed Slip-Flow, and Their Application to Several Specific Problems*, thesis, Calif. Inst. Tech., 1947

[4] Chang, C. S. Wang, and Uhlenbeck, G. E., *On the Transport Phenomena in Rarified Gases*, Applied Physics Lab. report No. APL/JHU CM-443, 1948.

ERRATA

In the article, "Multiply Valued Harmonic Functions. Green's Theorem," by G. C. Evans, these PROCEEDINGS, **33**, 270–275 (1947), Lemma 1, line 3, replace "*each*" by "$\mu(e)$;" and Lemma 2, line 3, replace "*lower semi-continuous*" by "*its limit inferior for approach from T.*"

alike, except that in some preparations the members of one pair appeared to be slightly curved while those of the other were straight.

The cross hermaphrodite × male showed seven diploid chromosomes. In the first anaphase three of these migrated to one pole while the remaining four moved in the opposite direction. The same was seen in preparations of asci derived from fertilization of a female by a hermaphrodite.

Thus a hermaphrodite haploid would have four chromosomes (one large, two medium and one small), while a male or a female would have only three (one large, one medium and one small). But how do the hermaphrodite and the neuter types arise from the cross female × male? In a few preparations it appeared that the six diploid chromosomes separated unequally, 4 going to one pole, only two to the other. This would mean that the two medium sized chromosomes, one of which contains the factor for male sex, the other for female, fail to separate in a certain number of cases (to judge from the genetical data) and both move to one pole. The spores formed from the nucleus containing only two chromosomes will be neuters.

The results obtained here provide an explanation for the derivation of male, female and even neuter clones from a hermaphroditic fungus. They also demonstrate that the haploid chromosome number in this fungus may be two, three or four.

These cytogenetic studies in Hypomyces are being continued by the writer.

* The work reported here was done under the direction of Prof. H. N. Hansen and Prof. W. C. Snyder of the Division of Plant Pathology, University of California, Berkeley, California.

[1] Hansen, H. N., and Snyder, W. C., "The Dual Phenomenon and Sex in *Hypomyces solani* f. *cucurbitae*," *Am. Jour. Bot.*, **30**, 419–422, 1943.

[2] Hansen, H. N., and Snyder, W. C., "Inheritance of Sex in Fungi," *Proc. Nat. Acad. Sci.*, **32**, 272–273, 1946.

MULTIPLY VALUED HARMONIC FUNCTIONS. GREEN'S THEOREM

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Communicated May 29, 1947

1. *The Multiple-Leaved Domain.*—A multiple-leaved space $\mathfrak{M}$ in three dimensions is the analog of a Riemann surface in the plane. Let T be a bounded domain on $\mathfrak{M}$, whose boundary consists of a finite number of closed branch curves $s_{(1)}, \ldots, s_{(\gamma)}$ and a bounded exterior frontier T^*, in

such a way that the situation is equivalent topologically to an m-leaved sphere in which the $s_{(i)}$ correspond to branch circles within the sphere, no two of which loop or have points in common. We assume that the $s_{(i)}$ are of zero capacity, considered as closed point sets in space. For our purposes there is no essential loss of generality if we restrict ourselves to a single branch curve s, connecting cyclically all m leaves. We are specially interested in the rôle of the branch curve, and since the boundary T^*, as described by means of the topologically equivalent image space, is not involved by it but lies on m separate leaves of $\mathfrak{M}$, we may take the parts $T_{(1)}^*, \ldots, T_{(m)}^*$ of the boundary to be regular surfaces.

THEOREM. *Let $u(M)$, $v(M)$ be two functions, bounded and harmonic in T and on T^*. Then, with n as the exterior normal to T^* at P,*

$$\int_{T^*}\left(u\frac{dv}{dn} - v\frac{du}{dn}\right)dP = 0, \tag{1}$$

$$\int_{T^*} u\frac{dv}{dn}dP = \int_T (\text{grad } u \cdot \text{grad } v)dM. \tag{2}$$

The essence of the theorem is that there is no contribution to the boundary integrals from the branch curves themselves. The proof of (1) is simpler than that of (2), although, of course, (1) is a consequence of (2).

In a paper which dealt primarily with multiple-valued Green's functions in the case where the branch curves were infinite straight lines, Sommerfeld[1] gave incidentally a proof of (2), assuming, however, that the branch curves were sufficiently smooth and that $u(M)$ was continuous on the branch curves. But this limitation is awkward and disguises the essential nature of the theorem. Examples show that continuity on the branch curves cannot be specified in advance.[2] It is essential, however, that the function be bounded and that the branch curves be sets of zero capacity, because if either of these restrictions is eliminated the theorem is no longer valid. The behavior at the branch curves tends to be determined by the geometric character of the domain rather than by the individual function.

The multiple-leaved space $\mathfrak{M}$ lies on a univalent base space, and the domain T lies on a univalent domain $\bar{T}$ composed of all points whose coördinates $(\bar{x}, \bar{y}, \bar{z})$ are the same as the coördinates of any point of T. We denote by barred symbols, in this way, the projection on the base space of a given configuration on $\mathfrak{M}$. A point Q of the branch curve s is defined to be a *limit point* of a set E of points on $T + s$, if $\bar{Q}$ is a limit point of $\bar{E}$.[3] With this definition, it is seen that any infinite set on $T + T^* + s$ has a limit point on $T + T^* + s$.

2. *Proof of (1)*.—The development of (1) may be indicated briefly. Construct a sequence of approximating domains T_k to T with boundaries T^* (fixed) and t_k (variable), the latter being multivalent tori which isolate

the s. The t_k may be obtained as images of tori in the equivalent topological space, and then smoothed out into regular surfaces by a well-known process. We may now determine a corresponding sequence of functions $u_k(M)$, $v_k(M)$, bounded and harmonic in T_k, and taking on the same values as $u(M)$, $v(M)$, respectively, on T^* and zero values on t_k. These functions are constructed by means of a generalization of the Schwarz alternating process. Since their behavior at a point of t_k or T^* depends merely on these boundaries locally, they are entirely regular at such points, and Green's theorem may be applied to T_k. Hence

$$\int_{T^*}\left(u_k\frac{dv_k}{dn} - v_k\frac{du_k}{dn}\right)dP + \int_{t_k}\left(u_k\frac{dv_k}{dn} - v_k\frac{du_k}{dn}\right)dP = 0.$$

But the last integral vanishes, through the definitions of u_k, v_k. Let then k become infinite. It is seen without difficulty that the u_k, v_k converge to functions $U(M)$, $V(M)$ harmonic in T and on T^*, and bounded, the convergence being uniform in the neighborhood of T^*. Hence

$$\int_{T^*}\left(U\frac{dV}{dn} - V\frac{dU}{dn}\right)dP = 0.$$

But also it is seen that U, V take on the same boundary values as u, v, respectively, on T^*. We shall see that they are identical with u, v in T, and in this way we shall obtain the identity (1).

In fact the function U-u vanishes on T^*, and is bounded and harmonic in T, and if it is not identically zero, either it or its negative is positive somewhere in T. Let $w(M)$ denote this choice of U-u or u-U. Then $w(M)$ has a positive upper bound B in T. We shall see that this is impossible on account of the hypothesis that s is of zero capacity.

3. *Kellogg's Theorem on the Upper Bound.*[4]—The theorem of O. D. Kellogg, on the capacity of sets on the boundary associated with the upper bound of a harmonic function, may be adapted to multiply valued functions, and, incidentally, applied to subharmonic functions.[5]

KELLOGG'S THEOREM. *Let $w(M)$ be subharmonic (in particular, harmonic) in the bounded domain T on $\mathfrak{M}$ and possess the finite least upper bound B in T. With $\epsilon > 0$, let e be the set of boundary points Q (that is, Q on T^* or on a branch curve) where*

$$\limsup_{M\to Q} w(M) \geq B - \epsilon,\ M \text{ in } T.$$

Then the base set $\bar{e}$ of e is closed and of positive capacity.

The proof follows closely the method of Kellogg. It will be noted that since the branch curves are of zero capacity the portion of e on the exterior boundary $T_{(i)}^*$ for at least one of the leaves $T_{(i)}$ must be of positive capacity. This fact insures the uniqueness of bounded harmonic functions in general as determined by their boundary values on T^*, without regard to their values on the branch curves, and in particular completes the proof of (1).

4. *Proof of (2).*—We consider first the summability over T of $(\text{grad } u)^2$ and the proof of the identity

$$\int_T (\text{grad } u)^2 dM = \int_{T^*} u \frac{du}{dn} dP. \tag{3}$$

The method used for (1) shows that

$$\begin{aligned}\int_{T^*} u\frac{du}{dn}dP &= \lim_{k\to\infty} \int_{T_k} (\text{grad } u_k)^2 dM \\ &= \int_{T_j} (\text{grad } u)^2 dM + \lim_{k\to\infty} \int_{T-T_j} (\text{grad } u_k)^2 dM \\ &= \lim_{j\to\infty} \int_{T_j} (\text{grad } u)^2 dM + \lim_{j\to\infty}\lim_{k\to\infty} \int_{T-T_j} (\text{grad } u_k)^2 dM,\end{aligned}$$

provided that we define u_k, for convenience, as zero in $T-T_k$. A similar identity holds for $\int_{T^*} u\frac{dv}{dn}dP$.

In particular the summability of $(\text{grad } u)^2$ is established, and since $2|\text{grad } u \cdot \text{grad } v| \leq (\text{grad } u)^2 + (\text{grad } v)^2$ the summability of $\text{grad } u \cdot \text{grad } v$ is also verified. Moreover if (3) is proved it will follow that

$$\lim_{j\to\infty}\lim_{k\to\infty} \int_{T-T_j} (\text{grad } u_k)^2 dM = 0,$$

so that the corresponding limit involving $\text{grad } u_k \cdot \text{grad } v_k$ will also vanish, and (2) will be proved.

Since we are dealing with a single branch curve s there is no loss of generality in assuming that all the branches $t_{(i)k}$ of the tori t_k have the same base set $\bar{t}_k$, and (by adjoining univalent domains) that all the branches $T_{(i)}{}^*$ of T^* have the same base set $\bar{T}^*$. The symmetric function,

$$V(\bar{M}) = \sum_{i=1}^{m} u_{(i)}{}^2(M),$$

will be bounded and univalent in $\bar{T}$. Moreover, the Laplacian of V will be given by the formula

$$\nabla^2 V(\bar{M}) = 2\sum_{i=1}^{m} (\text{grad } u_{(i)})^2,$$

$u_{(i)}(M)$ being harmonic, and $\nabla^2 V(\bar{M})$ will be summable and $V(\bar{M})$ subharmonic in T. Also $dV/dn = 2\Sigma_i u_{(i)} du_{(i)}/dn$. The identity (3) therefore is equivalent to the identity

$$\int_{\bar{T}} \nabla^2 V d\bar{M} = \int_{\bar{T}^*} \frac{dV}{dn} dP. \qquad (4)$$

It suffices to prove the latter.

We may now drop the bars in our notation, since henceforth we deal with univalent functions. The proof of (4) seems to involve considerable detail, but the main steps may be stated in terms of several lemmas. With $\rho(P) = \nabla^2 V(P)/(4\pi)$ and T' a domain which includes T (the erstwhile $\bar{T}$), but in which the properties of $V(M)$ still hold, we define the function

$$V_1(M) = V(M) + \int_{T'} \frac{\rho(P)}{MP} dP$$

which is harmonic in T' and bounded below, since $\rho(P) \geq 0$.

LEMMA 1. *Let F be a closed set of positive capacity and exterior frontier t, surrounded by a domain T of exterior boundary S, the latter being a closed regular surface. There exist two mass distributions $\mu(e)$ and $\nu(e)$, each of one sign, on t and S, respectively, whose combined potential takes on arbitrary constant values, namely, K at all points of S, and N at all points of t which are regular with respect to T. If $N \neq K$ the distribution $\mu(e)$ is not identically zero, and every point of t belongs to the nucleus of $\mu(e)$.*

The proof depends on considerations of minimum intrinsic energy.

LEMMA 2. *Let F and T be as in lemma 1 with the proviso, however, that F be of zero capacity. Let $V_1(M)$ be a function which is harmonic in T and bounded below, defined on F so as to be lower semi-continuous. Then the set $F + T$ is a domain T_1 and $V_1(M)$ is super-harmonic in T_1.* The proof of this lemma involves Kellogg's theorem, lemma 1 and the consideration of the functions $V_1^{(N)}(M)$, obtained by cutting off the function $V_1(M)$ by the arbitrary upper bound N.

We now apply lemma 2 to our problem, replacing F by s (the erstwhile $\bar{s}$), and T by T'. We make use of Riesz's theorem on superharmonic functions in order to deduce that in $T + s$, which is contained in $T' + s$, we may write $V_1(M) = p(M) + h(M)$, where $h(M)$ is harmonic and bounded and $p(M)$ is a potential of positive mass $\mu(e)$ on s. Hence, in $T + s$,

$$V(M) - h(M) = p(M) - \int_{T'} \frac{\rho(P)}{MP} dP, \qquad (5)$$

and the right hand member is bounded above.

With the aid again of Kellogg's theorem, this time as applied to univalent subharmonic functions, we can prove

LEMMA 3. *With T and s as above, and s of zero capacity, let $\mu(e)$ be a positive mass distribution on s, with $\mu(s) = 1$, and $\nu(e)$ a positive mass dis-*

tribution on T, with $\nu(T) = \nu < 1$. Then the difference of potentials $\int_s d\mu_p/MP - \int_T d\nu_p/MP$ is unbounded above.

In (5) the density $\rho(P)$ is summable. Hence the boundedness of its potential in the neighborhood of s depends merely on the values of ρ in a neighborhood of s. Accordingly, with respect to the boundedness of the right-hand member of (5), if $\mu(e)$ were not identically zero, we could discard temporarily as much of the distribution $\rho(P)$ as we pleased, outside that neighborhood, and assume that $\int_{T'}\rho(P)dP < \int_s d\mu_p$. But then, by lemma 3, the right-hand member of (5) would be unbounded above. Since this is not the case, we must have $\mu(e)$ identically zero. Hence

$$V(M) - h(M) = -\int_T \frac{\rho(P)}{MP}dP$$

for M in $T + s$. The function $\rho(P)$, according to its definition, is regular in the neighborhood of T^*. We may therefore apply Gauss's theorem to the univalent function $V(M) - h(M)$. Finally then

$$\int_{T^*}\frac{dV}{dn}dP = 4\pi\int_T \rho(M)dM = \int_T \nabla^2 V dM,$$

which is the equation (4) to be proved.

[1] Sommerfeld, A., "Uber verzweigte Potentiale im Raum," *Proc. London Math. Soc.* **28**, 395–429 (1897).

[2] Evans, G. C., "A necessary and sufficient condition of Wiener," *Amer. Math. Monthly*, **54**, 151–155 (1947).

[3] If there is more than one branch curve a subdomain $\mathfrak{Z}$ may be projected on the base space, in order to provide a definition of limit point.

[4] Kellogg, O. D., *Foundations of Potential Theory*, p. 335.

[5] A function is subharmonic in T if it is subharmonic in every univalent domain contained in T.

ON COMPLEXES OVER A RING AND RESTRICTED COHOMOLOGY GROUPS

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Communicated June 20, 1947

The relations between the fundamental group and the homology structure of a space S, developed recently by several authors,[1] may be formulated in a covering space of S as relations between the group of covering transformations (automorphisms) and "restricted" cohomology groups,