* This work was supported in part by the Office of Naval Research.

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[2] Weyl, H., "Inequalities Between the Two Kinds of Eigenvalues of a Linear Transformation," *Ibid.*, 35, 408–411 (1949).

[3] Pólya, G., "Remark on Weyl's Note: Inequalities Between the Two Kinds of Eigenvalues of a Linear Transformation," *Ibid.*, 36, 49–51 (1950).

A COMMUTATIVITY THEOREM FOR NORMAL OPERATORS

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Communicated by John von Neumann, November 9, 1949

1. *Introduction.*—This note contains two results concerning linear operators in Hilbert space $\mathfrak{H}$.

THEOREM I: *Let B be a bounded*[1] *operator and N a normal but not necessarily bounded operator with the canonical spectral representation*

$$N = \int z dE_z \qquad (z = x + iy).$$

Suppose that B commutes with N:

$$BN \subseteqq NB.$$

Then B commutes with E_z for any z: $BE_z = E_zB$ and hence B commutes with any function of N, e.g.

$$BN^* \subseteqq N^*B \qquad (\text{or } B^*N \subseteqq NB^*).$$

This theorem is very easily obtained in the case where N has a pure point spectrum. In the general case we may approximate N by operators with pure point spectra. Although these approximating operators in general do not commute with B, it turns out that the proof can be carried through along these lines, as shown in section 2. A second proof has later on been established by P. R. Halmos.

It is still an open question whether or not $NT \subseteqq TN$ implies $N^*T \subseteqq TN^*$ if N is bounded and normal and T is closed but non-bounded. In the case of two non-bounded operators the concept of commutativity is not even generally defined. It is, nevertheless, worth while to mention an example of two non-bounded, normal operators N_1 and N_2 which in a very suggestive way behave like commuting operators whereas N_1 and N_2^* behave like non-commuting operators. The example was constructed first by J. v. Neumann,[2] p. 61, footnote 37.

In section 3 we will discuss a conversion of Theorem I.

2. *Proof of Theorem I.*—We introduce an arbitrary square lattice with lines parallel to the coördinate axes of the complex plane. The length of

the sides of the squares is called s. The squares are considered as closed at left and below so that they are mutually disjoint. We arrange them as a sequence $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$. The center of σ_i is called z_i. For any Borel set α in the complex plane we denote by

$$E(\alpha) = \int_\alpha dE_z$$

the "spectral measure" of α. These projectors $E(\alpha)$ commute with N (and N^*). Evidently,

$$\Sigma_i E(\sigma_i) = I, \qquad E(\sigma_i)E(\sigma_k) = \delta_{ik}E(\sigma_k). \tag{1}$$

To any bounded operator A we attach a "matrix," the elements of which are operators: $A_{ik} = E(\sigma_i)AE(\sigma_k)$. It follows that

$$\Sigma_k A_{ik} = E(\sigma_i)A, \qquad \Sigma_i A_{ik} = AE(\sigma_k), \tag{2}$$

$$\Sigma_{i,\,k} A_{ik} = A. \tag{3}$$

Using the function: $h(z) = z_i$ when $z \in \sigma_i (i = 1, 2, \ldots)$, we introduce a normal operator

$$N' = h(N) = \int h(z)dE_z = \Sigma_i z_i E(\sigma_i).$$

N' has a pure point spectrum with the eigenvalues among the numbers z_i and the corresponding spectral manifolds are the ranges of the projectors $E(\sigma_i)$. As, moreover, N' commutes with any E_z, we get

$$E(\sigma_i)N' = N'E(\sigma_i) = z_i E(\sigma_i). \tag{4}$$

In order to estimate how well N' approximates N, we observe that $|z - h(z)| \leqq {}^1/_2 s\sqrt{2}$ (= the semidiagonal of each square). Hence the operator

$$N'' = \int \{z - h(z)\}dE_z$$

is bounded and

$$\|N''f\| \leqq {}^1/_2 s\sqrt{2}\|f\|, \qquad (f \in \mathfrak{H}). \tag{5}$$

Thus we have (see, e.g., Nagy,[3] p. 45, property d)

$$N' + N'' = \int h(z)dE_z + \int \{z - h(z)\}dE_z = \int z dE_z = N, \tag{6}$$

and the domains of N and N' are equal, say $\mathfrak{D}$.

Now, let f denote an arbitrary element of $\mathfrak{D}$. Since $BN \subseteq NB$ even Bf belongs to $\mathfrak{D}$ and we have by assumption $BNf = NBf$. Using equation (6) and the fact that N'' is everywhere defined, we obtain

$$BN'f - N'Bf = -BN''f + N''Bf.$$

Replacing f by $E(\sigma_k)f$ which does belong to $\mathfrak{D}$ and applying $E(\sigma_i)$ on both sides of this equation, we get

$$E(\sigma_i)BN'E(\sigma_k)f - E(\sigma_i)N'BE(\sigma_k)f = -E(\sigma_i)BN''E(\sigma_k)f + E(\sigma_i)N''BE(\sigma_k)f,$$

i.e.,

$$(z_k - z_i)B_{ik}f = -B_{ik}N''f + N''B_{ik}f$$

by use of equation (4) and (on the right-hand side) $E(\sigma_i)N'' = N''E(\sigma_i)$. From this and relation (5) we derive an estimation which is the central point of the proof:

$$|z_k - z_i| \, ||B_{ik}f|| \leqq {}^1/_2 s\sqrt{2}|||B_{ik}||| \, ||f||. \tag{7}$$

(For a bounded operator A we denote by $|||A|||$ the greatest lower bound for the numbers c for which $||Af|| \leqq c||f||$ for all f). By continuity, equation (7) remains true for any $f \in \mathfrak{H}$, whence for $i \neq k$

$$|||B_{ik}||| \leqq \frac{s\sqrt{2}}{|z_k - z_i|} |||B_{ik}|||. \tag{8}$$

This inequality permits us to conclude that

$$|||B_{ik}||| = 0, \qquad \text{i.e., } B_{ik} = 0, \tag{9}$$

for any i, k for which

$$|z_k - z_i| > s\sqrt{2}. \tag{10}$$

The only pairs i, k for which the condition (10) is not fulfilled are such where σ_i and σ_k are neighbors in the sense that they touch each other either at a corner or along a side. Even in this case we can, however, prove that $B_{ik} = 0$ holds if $i \neq k$. We divide each side of each square σ_i into n equal parts in order to introduce a new lattice, n times as fine as the original one. Consider now our two neighbors σ_i and σ_k, and let them, e.g., touch along a horizontal side, σ_i being just below σ_k. By the subdivision we get n^2 small squares $\sigma_i^p(p = 1, 2, \ldots, n^2)$ inside σ_i and n^2 small squares $\sigma_k^q(q = 1, 2, \ldots, n^2)$ inside σ_k. Then

$$E(\sigma_i) = \sum_{p=1}^{n^2} E(\sigma_i^p), \qquad E(\sigma_k) = \sum_{q=1}^{n^2} E(\sigma_k^q)$$

and hence

$$B_{ik} = E(\sigma_i)BE(\sigma_k) = \sum_{p,q=1}^{n^2} E(\sigma_i^p)BE(\sigma_k^q). \tag{11}$$

According to the above result (9), but now applied to the new, finer lattice, those terms in equation (11) which correspond to non-neighbors in the fine lattice vanish. Thus we are allowed to consider only those terms $E(\sigma_i^p)BE(\sigma_k^q)$ where σ_i^p "touches" the large square σ_k (whereas σ_k^q may be any of the small squares inside σ_k). Denoting by ρ_n the small rectangle composed of exactly these latter σ_i^p, we may recollect the terms thus considered:

$$B_{ik} = \sum_{\sigma_i^p \subseteqq \rho_n,\ q = 1,2,\ldots,n^2} E(\sigma_i^p)BE(\sigma_k^q) = E(\rho_n)BE(\sigma_k). \tag{12}$$

This equation being valid for any natural number n, we may pass to the limit $n \to \infty$, whereby the rectangles ρ_n decrease toward their intersection which is empty because we have considered the lattice squares as "semi-closed" in the previously indicated way. By the total additivity (or "multiplicativity") of the spectral measure $E(\alpha)$ we therefore obtain

$$\lim_{n \to \infty} E(\rho_n) = 0,$$

and hence from equation (12) the desired result $B_{ik} = \lim_{n \to \infty} E(\rho_n)BE(\sigma_k) = 0$. Thus we have proved the equation

$$B_{ik} = E(\sigma_i)BE(\sigma_k) = 0 \quad \text{for} \quad i \neq k. \tag{13}$$

In order to show that B commutes with E_z for any given complex number z, we choose the original square lattice so that the point $z = x + iy$ is a lattice point. We denote by ζ the quarterplane consisting of all points with real part $<x$ and with imaginary part $<y$. This quarterplane is a sum of squares σ_i from our lattice. Now

$$E_z = E(\zeta) = \Sigma_i' E(\sigma_i)$$

where the apostrophe denotes that the summation is to be restricted to squares $\sigma_i \subseteqq \zeta$. Thus we get, remembering equation (1),

$$E_zB = \Sigma_i' E(\sigma_i)B = \Sigma_i' \Sigma_k E(\sigma_i)BE(\sigma_k).$$

On account of equation (13) this gives

$$E_zB = \Sigma_i' E(\sigma_i)BE(\sigma_i).$$

Next, when computing BE_z in a similar manner, we obtain exactly the same result and we have thus proved that

$$BE_z = E_zB.$$

The rest of Theorem I follows easily from this relation by use of the operational calculus (see, e.g., Nagy,[3] p. 45, property b, and p. 29, top of page).

3. On a Conversion of Theorem I.—I. E. Segal has kindly drawn my attention to a slightly different formulation of Theorem I: *Any normal operator N has the property P that the ring of all bounded operators B commuting with N is a self-adjoint ring;* that is, whenever the ring contains B then it contains also B^*. We may ask whether this property P *characterizes* the class of normal operators. This is actually the case if we consider *bounded* operators only. For if N is a bounded operator with the above property P, then we simply choose $B = N$ and infer that $NN^* =$

N^*N, q. e. d. Next let a *non-bounded* operator T have the property P. Here we cannot choose $B = T$. This might seem to be a mere technical difficulty, but that is actually not the case as shown by the following theorem.

THEOREM II. *There exists a closed operator T (with a domain everywhere dense in $\mathfrak{H}$) which does not commute with any bounded operator, except with the numerical multiples of the identity I.*

The operator T, constructed below as an example proving this theorem, has, furthermore, the entire complex plane as point spectrum, any complex number being a simple eigenvalue. T is, of course, not normal.

In the Hilbert space $\mathfrak{H} = \mathfrak{L}^2(-\infty < x < \infty)$ we consider the self-adjoint operators[4] $P = -id/dx$ and $Q = x\cdot$ and form the operator $T = Q + iP$, which is defined in a dense set $\mathfrak{D}$. For a function $f(x) \in \mathfrak{D}$ we have $Tf(x) = xf(x) + f'(x)$. In order first to prove that T is closed, we consider any sequence $\{f_n\}$ $(f_n \in \mathfrak{D})$ for which $\lim_n f_n (= f \in \mathfrak{H})$ and $\lim_n Tf_n (= g \in \mathfrak{H})$ exist. We then have to prove that $f \in \mathfrak{D}$ and that $Tf = g$. It is sufficient to show the existence of $\lim_n Qf_n$ and $\lim_n Pf_n$, for this implies (Q and P being closed) that $f \in \mathfrak{D}_Q \cap \mathfrak{D}_P = \mathfrak{D}$ and that $\lim_n Qf_n = Qf$, $\lim_n Pf_n = Pf$, whence $\lim_n Tf_n = \lim_n Qf_n + i \lim_n Pf_n = Qf + iPf = Tf$, q. e. d. Now, let h be an arbitrary element of $\mathfrak{D}$. Then

$$||(Q + iP)h||^2 = ||Qh||^2 + ||Ph||^2 + (Qh, iPh) + (iPh, Qh), \qquad (14)$$

$(Qh, iPh) + (iPh, Qh) = \int_{-\infty}^{\infty} xh(x)\overline{h'(x)}\, dx + \int_{-\infty}^{\infty} h'(x)\overline{xh(x)}\, dx = \int_{-\infty}^{\infty} x(d/dx)|h(x)|^2\, dx = [x|h(x)|^2]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} |h(x)|^2\, dx \geqq 0 - ||h||^2$. Substituting in equation (14), we get $||Qh||^2 + ||Ph||^2 \leqq ||h||^2 + ||Th||^2$, by which inequality the convergence of $\{Qf_n\}$ and of $\{Pf_n\}$ is derived from that of $\{f_n\}$ and of $\{Tf_n\}$ putting $h = f_m - f_n$ and making $m, n \to \infty$. Thus T is a closed operator.

If s is any complex number, then the equation $Tf = sf$ may be written $f'(x) + (x - s)f(x) = 0$, the only solutions of which are $f(x) = \text{const.}\, f_s(x)$, where

$$f_s(x) = \exp\{-{}^1/_2(x - s)^2\} \in \mathfrak{D}.$$

Thus s is a simple eigenvalue for T, the corresponding eigenelements being cf_s where c is an arbitrary complex number. Suppose now that a bounded operator B commutes with $T: BT \subseteqq TB$. In particular, $TBf_s = BTf_s$ for any s; that is $TBf_s = sBf_s$, showing that Bf_s (if $\neq 0$) is an eigenelement for T belonging to the eigenvalue s; i.e.,

$$Bf_s = c_s f_s, \qquad (15)$$

where c_s is a certain number depending on s only. On account of the regular way in which f_s depends on s, combined with the boundedness of B, we may show that c_s is a *differentiable* (i.e., analytic) function of s

in the whole s-plane, and as c_s is *bounded* (because B is bounded) we infer by Liouville's Theorem that c_s is a *constant* c. The details may be carried out in the following way. First we prove that

$$(s - s')^{-1}(f_s - f_{s'}) \rightarrow g_{s'} \qquad \text{(in } \mathfrak{H}\text{) as } s \rightarrow s',\ s \neq s', \tag{16}$$

where $g_s(x) = \partial f_s(x)/\partial s = (x - s) \exp \{- {}^1/_2 (x - s)^2\}$. The relation (16) holds, of course, in the sense of pointwise convergence. Now we may show that $|f_s(x) - f_{s'}(x)| \leq |g_{s''}(x)|/|s-s'|$ where $s'' \rightarrow s'$ as $s \pm s'$. An estimation of $|g_{s''}(x)|$ justifies the application of Lebesgue's convergence theorem whereby relation (16) is proved. An immediate consequence of this result (16) is that the complex function (f_s, h) of s is differentiable at any s; h being any fixed element of $\mathfrak{H}$. In order to prove that c_s is differentiable at any given point s' we use equation (15) to write

$$c_s(f_s, f_{s'}) = (c_s f_s, f_{s'}) = (Bf_s, f_{s'}) = (f_s, B^* f_{s'}).$$

Here $(f_s, f_{s'})$ and $(f_s, B^*f_{s'})$ are both differentiable (choose $h = f_{s'}$ and $= h$ $B^*f_{s'}$, respectively). Hence $c_s = (f_s, B^*f_{s'})/(f_s, f_{s'})$ is differentiable at any point s where $(f_s, f_{s'}) \neq 0$, in particular at $s = s'$, q. e. d.

From $Bf_s = cf_s$ for all s we finally conclude that

$$Bf = cf \qquad \text{for every } f \in \mathfrak{H}, \tag{17}$$

considering that B is a bounded (i.e., continuous) operator and that the set of all finite linear combinations of the elements f_s is *everywhere dense* in $\mathfrak{H}$. In fact this latter statement holds even by restricting s to take real values only, as we may show, e.g., by use of the Fourier integral calculus. Suppose that an element $h \in \mathfrak{H}$ is orthogonal to every f_s (s real):

$$(h, f_s) = \int_{-\infty}^{\infty} h(x) \exp \{-{}^1/_2(s - x)^2\} dx = 0 \text{ for all } s. \tag{18}$$

This means that the convolution $h(x) * \exp(- {}^1/_2 x^2)$ of the two $\mathfrak{L}^2$-functions $h(x)$ and $\exp(- {}^1/_2 x^2)$ is identically zero. But this implies that the product of the Fourier transforms of $h(x)$ and of $\exp(-{}^1/_2 x^2)$ vanishes. As the latter Fourier transform is $\exp(- {}^1/_2 x^2) \neq 0$ we infer that $h(x) = 0$. This means, however, that the set $\{f_s\}$ spans $\mathfrak{H}$. We have now established all the properties of T announced in Theorem II and the succeeding remarks.

[1] The expression *bounded* operator means in this note a bounded, linear operator which is defined in the entire Hilbert space.

[2] *Portugaliae Math.*, **3**, 1–62 (1942), particularly appendix 3, pp. 60–61.

[3] Nagy, Béla Sz., "Spektraldarstellung linearer Transformationen des Hilbertschen Raumes," *Ergeb. d. Math.*, V, 5, Berlin, 1942.

[4] Stone, M. H., *Linear Transformations in Hilbert Space*, Am. Math. Soc. Coll. Publ. XV, New York, 1932. About the self-adjoint operator $-id/dx$, see Theorem 10.9.