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CONCERNING NON-CONTINUABLE, TRANSCENDENTALLY TRANSCENDENTAL POWER SERIES

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Communicated by J. L. Walsh, February 23, 1951

The main purpose of this note is to show that power series of the kind described in the title can be obtained from a given power series by simply multiplying certain of its coefficients by -1.

Consider the class $\mathcal{K}$ of power series of the form $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ whose circle of convergence is the unit circle. There are $\mathfrak{c}$ elements in $\mathcal{K}$ (where $\mathfrak{c}$ denotes the power of the continuum). Let $\mathcal{C}$ be the class of those series in $\mathcal{K}$ which can be continued beyond the unit circle, and let $\mathcal{A}$ be the class of those series in $\mathcal{K}$ which satisfy an algebraic differential equation. Denote by $\mathcal{C}'$, $\mathcal{A}'$, the respective complements of $\mathcal{C}$, $\mathcal{A}$, with respect to $\mathcal{K}$.

There are the following sufficient conditions for a series in $\mathcal{K}$ to belong to $\mathcal{C}'$, $\mathcal{A}'$, respectively:

(A)[1] *Let* $\{\lambda_\nu\}$ $(\nu = 1, 2, 3, \ldots)$ *be an increasing sequence of non-negative integers such that* $\lambda_\nu/\nu \to \infty$ *as* $\nu \to \infty$. *If* $\sum_{\nu=1}^{\infty} a_\nu z^{\lambda_\nu}$ *belongs to* $\mathcal{K}$, *then it also belongs to* $\mathcal{C}'$.

(B)[2] *Let* $\{\lambda_\nu\}$ $(\nu = 1, 2, 3, \ldots)$ *be a sequence of non-negative integers such that* $\lambda_{\nu+1} > \nu\lambda_\nu$ *for every* ν. *If* $\sum_{\nu=1}^{\infty} a_\nu z^{\lambda_\nu}$ *belongs to* $\mathcal{K}$, *then it also belongs to* $\mathcal{A}'$.

The series $\sum_{\nu=0}^{\infty} z^\nu$, which represents $(1 - z)^{-1}$ for $|z| < 1$, belongs to $\mathcal{C}\mathcal{A}$ (i.e., to both $\mathcal{C}$ and $\mathcal{A}$). The series $\sum_{\nu=0}^{\infty} b_\nu z^\nu$, which represents the meromorphic function $\Gamma(z + 1)$ for $|z| < 1$, belongs to $\mathcal{C}$ and[3] to $\mathcal{A}'$. According to (A), $\sum_{\nu=0}^{\infty} z^{\nu^2}$ belongs to $\mathcal{C}'$, and it is known[4] that this series belongs to $\mathcal{A}$. Finally, it follows from (A) and (B) that $\sum_{\nu=0}^{\infty} z^{\nu!}$ belongs to $\mathcal{C}'\mathcal{A}'$. Thus, none of the classes $\mathfrak{C}\mathfrak{A}$, $\mathfrak{C}\mathfrak{A}'$, $\mathfrak{C}'\mathfrak{A}$, $\mathfrak{C}'\mathfrak{A}'$ is empty. In fact, *each of these classes contains* $\mathfrak{c}$ *elements;* for if an arbitrary constant is added to any one of the four series just mentioned, the resulting series belongs to the same class as the original.

Let $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ belong to $\mathfrak{K}$. Call a sequence $\{\epsilon_\nu\}$ such that $\epsilon_\nu = \pm 1$ ($\nu = 0, 1, 2, \ldots$) a "sign sequence." Then[5] there is a sign sequence $\{\epsilon_\nu\}$ such that $\sum_{\nu=0}^{\infty} \epsilon_\nu a_\nu z^\nu$ belongs to $\mathfrak{C}'$; indeed, there are $\mathfrak{c}$ such sign sequences.[6] It is also known[7] that there are infinitely many sign sequences $\{\delta_\nu\}$ such that $\sum_{\nu=0}^{\infty} \delta_\nu a_\nu z^\nu$ belongs to $\mathfrak{A}'$.[8] We now prove the following

THEOREM. *Let* $\sum_{\nu=0}^{\infty} a_\nu z^\nu = f(z)$ *belong to* $\mathfrak{K}$. *Then there are* $\mathfrak{c}$ *sign sequences* $\{\epsilon_\nu\}$ *such that* $\sum_{\nu=0}^{\infty} \epsilon_\nu a_\nu z^\nu$ *belongs to* $\mathfrak{C}'\mathfrak{A}'$.

Proof:[9] Since our series, by assumption, belongs to $\mathfrak{K}$, there is an increasing sequence of natural numbers $\{\nu_\kappa\}$ ($\kappa = 1, 2, 3, \ldots$) such that, for every κ, $\nu_{\kappa+1} > \kappa\nu_\kappa$, $a_{\nu_\kappa} \neq 0$, and such that $\lim |a_{\nu_\kappa}|^{1/\nu_\kappa} = 1$ as $\kappa \to \infty$. It follows then that $\nu_\kappa/\kappa \to \infty$ as $\kappa \to \infty$. According to (A) and (B), $\sum_{\kappa=1}^{\infty} a_{\nu_\kappa} z^{\nu_\kappa} = g(z)$ belongs to $\mathfrak{C}'\mathfrak{A}'$; and the series $a \cdot g_{\mu\sigma}(z) = a \cdot \sum_{\kappa=1}^{\infty} \sigma_\kappa a_{\mu_\kappa} z^{\mu_\kappa}$, where $a \neq 0$ is an arbitrary constant, $\{\mu_\kappa\}$ is any infinite subsequence of $\{\nu_\kappa\}$, and $\{\sigma_\kappa\}$ is any sign sequence, belongs *a fortiori* to $\mathfrak{C}'\mathfrak{A}'$. Set $f(z)$-$g(z)$ $= f_0(z)$. Divide $g(z)$ into an infinite sequence of power series $f_1(z)$, $f_2(z)$, $\ldots$, $f_\rho(z)$, $\ldots$, each of which contains infinitely many terms, such that every term of $g(z)$ belongs to precisely one $f_\rho(z)$ with $\rho \geqq 1$. Consider the set of all power series

$$F(z) = f_0(z) + \sum_{\gamma=1}^{\infty} \delta_\gamma f_\gamma(z), \qquad \delta_\gamma = \pm 1. \tag{1}$$

At most an enumerable number of these series can belong to $\mathfrak{C}$.[6] Hence, $\mathfrak{c}$ of them must belong to $\mathfrak{C}'$, and these can be divided into $\mathfrak{c}$ pairs, because $\mathfrak{c} = \mathfrak{c} + \mathfrak{c}$. If $F_1(z)$ and $F_2(z)$ are the members of any one of these pairs, then it is evident from (1) that $F_1(z) - F_2(z) = 2g_{\mu\sigma}(z)$ for suitable sequences μ and σ. As we remarked before, $2g_{\mu\sigma}(z)$ belongs to $\mathfrak{A}'$, so that[10] at least one of $F_1(z)$, $F_2(z)$ belongs to $\mathfrak{A}'$; and this completes the proof.

The theorem does not remain valid if $\mathfrak{C}'\mathfrak{A}'$ is replaced by any one of the other three classes, because, according to (A) and (B), $\sum_{\nu=0}^{\infty} \epsilon_\nu z^{\nu!}$, e.g., belongs to $\mathfrak{C}'\mathfrak{A}'$ for every sign sequence $\{\epsilon_\nu\}$, due to the presence of large gaps (i.e., consecutive terms whose coefficients are zero). *There are, moreover,* $\mathfrak{c}$ *series in* $\mathfrak{K}$, *none of which has gaps, and yet each of which belongs to* $\mathfrak{C}'\mathfrak{A}'$ *for every sign sequence.* For let $\{\epsilon_\nu\}$ be an arbitrary sign sequence. Consider

$\sum_{\nu=0}^{\infty}\epsilon_\nu a_\nu z^\nu$, where $a_0 = 1$, $a_\nu = \nu$ for every ν belonging to the sequence $\{\nu_\kappa\}$, where $\nu_1 = 1$, $\nu_{\kappa+1} = \nu_\kappa + \kappa$ ($\kappa = 1, 2, 3, \ldots$), whereas $a_\nu = 1/\nu^{\nu^2}$ for every other ν (this series obviously belongs to $\mathcal{K}$). Every coefficient is different from zero, and is an algebraic number, being either an integer or the reciprocal of an integer. Furthermore, there is clearly no constant $c > 0$ such that $|\epsilon_\nu/\nu^{\nu^2}| \geqq \exp(-c\nu(\log \nu)^2)$ for every $\nu \geqq 2$. Consequently,[11] $\sum_{\nu=0}^{\infty}\epsilon_\nu a_\nu z^\nu$ belongs to $\mathcal{A}'$. If ν is not a term of the sequence $\{\nu_\kappa\}$, then $|\epsilon_\nu a_\nu| \leqq 1$. The sequence $\{\epsilon_{\nu_\kappa} a_{\nu_\kappa}\}$, however, is unbounded, and $\nu_{\kappa+1} - \nu_\kappa \to \infty$ as $\kappa \to \infty$. Therefore[12] $\sum_{\nu=0}^{\infty}\epsilon_\nu a_\nu z^\nu$ cannot be bounded in any sector of the unit circle, which means that this series belongs to $\mathcal{C}'$. Thus, $\sum_{\nu=0}^{\infty}\epsilon_\nu a_\nu z^\nu$ has no gaps and yet belongs to $\mathcal{C}'\mathcal{A}'$ for every sign sequence $\{\epsilon_\nu\}$. We can easily obtain $\mathfrak{c}$ such series, and, in fact, having rational coefficients, from $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ by replacing $a_\nu = 1/\nu^{\nu^2}$, for every $\nu \neq 0$ which does not belong to $\{\nu_\kappa\}$, by any one of the numbers $1/\tau^{\tau^2} (\tau = \nu, \nu + 1, \nu + 2, \ldots)$.

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