[4] In a paper presented by Anne Whitney at the International Congress of Mathematicians, September, 1950.

[5] This is a discrete analog of a problem solved by Schoenberg; see the paper "On Totally Positive Functions, Laplace Integrals and Entire Functions of the Laguerre-Polya-Schur Type," PROC. NATL. ACAD. SCI., **33**, 11–17 (1947).

[6] For the special case when $f(z)$ is entire, hence of the form (6), this Theorem 7 is due to Polya; see Polya, G., "Über einen Satz von Laguerre," *Jahresb. der Deutschen Mathematiker-Vereinigung*, **38**, 161–168 (1929), especially pages 166–168.

[7] Laguerre, "Sur les fonctions du genre zéro et du genre un," *Oeuvres de Laguerre*, Vol. 1, Paris, 1898, pp. 174–177, and Polya, G., "Über Annäherung durch Polynome mit lauter reellen Wurzeln," *Rendiconti di Palermo*, **36**, 1–17 (1913).

COHOMOLOGY THEORY OF ABELIAN GROUPS AND HOMOTOPY THEORY III

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1. *Introduction.*—For each abelian group Π and for each integer n we consider the complexes $K(\Pi, n)$ described in Note I[1] and the suspension maps $S:K(\Pi, n) \rightarrow K(\Pi, n + 1)$ raising the dimensions by 1. We recall that $K(\Pi, n)$ gives the singular homology and cohomology theory for any arcwise connected space X in which only the nth homotopy group $\pi_n(X) = \Pi$ is non-trivial ($n > 1$). We show now that the augmented complex $K(\Pi, n)$ is, after a change in dimension, chain equivalent to the complex $A^{n-1}(\Pi)$ which gives the "abelian" homology theory of Π in level $n - 1$. This proves the main conjecture (26) of Note II.[1]

2. *Products.*—The complex $K(\Pi, n)$ is a complete semisimplicial complex.[2] This means the following. If $[q]$ denotes the set of integers i with $0 \leqq i \leqq q$ and $\alpha : [p] \rightarrow [q]$ any weakly monotone map, each q-simplex σ_q of $K(\Pi, n)$ determines a p-simplex $\sigma_q\alpha$, with appropriate formal properties. In particular, the ith face of σ_q is the $(q - 1)$-simplex $F_i\sigma_q = \sigma_q\epsilon^i$, where $\epsilon^i : [q - 1] \rightarrow [q]$ is the monotone map covering all of $[q]$ except the integer i. We also use the degenerate simplices $N_i\sigma_q = \sigma_q\eta^i$, defined for $0 \leqq i \leqq q$ by the maps $\eta^i = \eta_q^i$: $[q + 1] \rightarrow [q]$ with $\eta^i(j) = j$, for $j \leqq i$ and $\eta^i(j) = j - 1$ for $i < j$. We denote by $K'(\Pi, n)$ the quotient of $K(\Pi, n)$ by the subcomplex consisting of all degenerate cells and the single 0-cell. If $K_a(\Pi, n)$ is the complex $K(\Pi, n)$ augmented in dimension -1 by the group of integers, then the natural map $K_a(\Pi, n) \rightarrow K'(\Pi, n)$ is a chain equivalence, by a known normalization theorem.[2] The suspension map $S:K'(\Pi, n) \rightarrow K'(\Pi, n + 1)$ is still valid.

Because of the addition in Π we may construct for any two q-simplices σ and τ in $K(\Pi, n)$ a new q-simplex $\sigma \oplus \tau$, their *internal sum*, defined as the sum of σ and τ, considered (Note I) as elements of $Z^n(\Delta_q, \Pi)$. Using this sum, we define two products $*$ and $\downarrow$, as follows. For any non-negative integers p and q consider all partitions of the set $[p + q - 2]$ into two disjoint sets a and b, with $q - 1$ and p elements, respectively. Write the integers in the set a in descending order as $a = \{a_{q-1}, \ldots, a_1\}$ and set

$$N_a = N_{a_{q-1}} \ldots N_{a_1}, \qquad \epsilon(a) = \sum_{j=1}^{q-1} a_j - (j - 1). \tag{1}$$

For simplices σ_p and τ_q of dimensions p and q, respectively, in $K(\Pi, n)$ a chain $\sigma_p \downarrow \tau_q$ of dimension $p+q$ is now defined as

$$\sigma_p \downarrow \tau_q = \sum (-1)^{\epsilon(b)} [N_a N_p \sigma_p \oplus N_b \tau_q], \tag{2}$$

with the sum taken over all partitions a, b as above. The formula

$$\sigma_p * \tau_q = \sigma_p \downarrow \tau_q + (-1)^{pq} \tau_q \downarrow \sigma_p \tag{3}$$

now yields a product of excess zero, in the sense of Note II. It has a "geometric" interpretation in terms of the triangulation of the cartesian product $\Delta_p \times \Delta_q$ of standard affine simplices. Both these products carry over to $K'(\Pi, n)$, and $*$ is still a product of excess zero there.

3. *The Main Transformation.*—For each cell complex L with a product of excess 0, we have defined in Note II, by means of the so-called "bar construction," a complex $\mathcal{B}(L)$ with a product of excess 1, containing L as a subcomplex. We shall denote by $\mathcal{B}^+(L)$ the complex obtained from $\mathcal{B}(L)$ by raising all dimensions one unit. Then in $\mathcal{B}^+(L)$ the product has excess 0, while the inclusion map $L \rightarrow \mathcal{B}(L)$ becomes a map $S_B: L \rightarrow \mathcal{B}^+(L)$ raising the dimension by 1.

We now define a chain transformation

$$f: \mathcal{B}^+(K'(\Pi, n)) \rightarrow K'(\Pi, n + 1). \tag{4}$$

Specifically, if σ is a p-simplex of $K'(\Pi, n)$, and hence a $p + 1$ simplex of $\mathcal{B}^+(K'(\Pi, n + 1))$, we set $f(\sigma) = S(\sigma)$. Any other cell of the complex on the left has one or more bars, and thus may be written as $[\rho | \sigma]$, where ρ is a cell with one fewer bar, and σ a cell of $K'(\Pi, n)$. We thus define f by induction on the number of bars as

$$f[\rho|\sigma] = f\rho \downarrow f\sigma, \qquad f\sigma = S(\sigma). \tag{5}$$

(The product $\downarrow$ is not associative.)

THEOREM 1. *The map f of (4) is a chain equivalence, with the additional properties*

$$f(\rho_1 * \rho_2) = f(\rho_1) * f(\rho_2), \tag{6a}$$

$$fS_B = S. \tag{6b}$$

The proof depends upon a representation of $K(\Pi, n + 1)$ in terms of $K(\Pi, n)$. Specifically, any $q + 1$ cell of $K(\Pi, n + 1)$ can be represented uniquely as a $(q + 1)$-tuple

$$\tau_{q+1} = \langle \sigma_q, \sigma_{q-1}, \ldots, \sigma_0 \rangle, \tag{7}$$

where each σ_i is an i-cell of $K(\Pi, n)$. The semisimplicial structure of $K(\Pi, n + 1)$ is determined by giving the operator for the ith face F_i, as

$$\begin{aligned} F_0\tau_{q+1} &= \langle F_0\sigma_q, \ldots, F_0\sigma_1 \rangle, \\ F_i\tau_{q+1} &= \langle F_i\sigma_q, \ldots, F_i\sigma_{i+1}, F_i\sigma_i \oplus \sigma_{i-1}, \sigma_{i-2}, \ldots, \sigma_0 \rangle, \\ F_{q+1}\tau_{q+1} &= \langle \sigma_{q-1}, \ldots, \sigma_0 \rangle, \end{aligned}$$

for $0 < i < q + 1$, and the operator for the ith "degenerate" simplex as

$$N_i\tau_{q+1} = \langle N_i\sigma_q, \ldots, N_i\sigma_i, 0_i, \sigma_{i-1}, \ldots, \sigma_0 \rangle,$$

with $0 \leqq i \leqq q + 1$, where 0_i denotes the cell (cocycle) which is identically zero.

By means of this representation, we introduce in $K'(\Pi, n + 1)$ a family of subcomplexes K_k', spanned by those cells τ with at most k non-zero entries in (7). In $\mathcal{B}^+(K(\Pi, n)F)$ we use the family of subcomplexes $\mathcal{B}_k^+$, spanned by the cells with less than k bars ($k = 1, 2, \ldots$). The map f of (4) carries $\mathcal{B}_k^+$ into K_k', and the proof of the Theorem is completed by showing that each of the induced maps for the factor complexes $\mathcal{B}_{k+1}^+/\mathcal{B}_k^+ \rightarrow K_{k+1}'/K_k'$ is a chain equivalence.

The proof of Theorem 1 also is independent of the value of the integer n, and actually yields a more general theorem as to the equivalence of the processes $\mathcal{B}^+$ and (7), when applied to a complete semisimplicial complex which has a suitable internal sum and which has only one cell in dimension zero.

4. *The Abelian Homology Theory.*—Let $A(\Pi, 1)$ denote the complex $K(\Pi, 1)$ with the single 0-simplex removed. We define two sequences of complexes $A(\Pi, n)$, $A'(\Pi, n)$, as follows

$$A(\Pi, n + 1) = \mathcal{B}^+(A(\Pi, n)), \tag{8}$$

$$A'(\Pi, 1) = K'(\Pi, 1), \qquad A'(\Pi, n + 1) = \mathcal{B}^+(A'(\Pi, n)). \tag{9}$$

These complexes all have products (of excess 0) and maps $S_A : A(\Pi, n) \rightarrow A(\Pi, n + 1)$ and $S_A' : A'(\Pi, n) \rightarrow A'(\Pi, n + 1)$, raising the dimension by 1. If in $A(\Pi, n)$ all dimensions are lowered by $n - 1$, there results the complex $A^{n-1}(\Pi)$ of Note II.[3] In virtue of the normalization theorem quoted above, there is a chain equivalence $\eta : A(\Pi, 1) \rightarrow A'(\Pi, 1)$, hence, by the iteration process described in Theorem 2 of Note II[4] there are chain equivalences $\eta_n : A(\Pi, n) \rightarrow A'(\Pi, n)$, which preserve the products and commute with the suspensions S_A. Using Theorem 1 above and Theorem 2 of Note II we obtain

THEOREM 2. *Iteration on the map f of (4) yields maps*

$$f_n : A'(\Pi, n) \to K'(\Pi, n)$$

which are chain equivalences, satisfy $f_n(\sigma_* \tau) = f_n(\sigma)_* f_n(\tau)$, *and yield a commutative diagram*

$$\begin{array}{ccccccc} \ldots \to & A'(\Pi, n) & \xrightarrow{S^{+A}} & A'(\Pi, n+1) & \to \ldots \\ & \downarrow f_n & S & \downarrow f_{n+1} & \\ \ldots \to & K'(\Pi, n) & \longrightarrow & K'(\Pi, n+1) & \to \ldots . \end{array}$$

This result has as a corollary the suspension Theorem (Theorem 1) of Note I. It also proves that the conjecture expressed in formula (26) of Note II is correct.

The homology and cohomology groups of $K(\Pi, n)$ over a coefficient group G are denoted by $H_q(\Pi, n; G)$ and $H^q(\Pi, n; G)$. The groups derived from the complexes $A(\Pi, n)$ have, as shown in Note II, a direct connection with the so-called homology theory of abelian groups.[5] Theorem 2 yields isomorphisms

$$H_q(\Pi, n; G) \cong H_q(A(\Pi, n); G), \quad q > 0,$$
$$H^q(\Pi, n; G) \cong H^q(A(\Pi, n); G), \quad q > 0,$$

without the use of the intermediate cubical complexes $Q_t(\Pi)$ of Note I. When Π and G are finitely generated these groups are also finitely generated (and finitely computable), as observed by Serre.[6]

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[1] Eilenberg, S., and MacLane, S., PROC. NATL. ACAD. SCI., **36**, 443–447 and 657–663 (1950); subsequently referred to as Note I and Note II.

[2] Eilenberg, S., and Zilber, J. A., *Ann. Math.*, **51**, 499–513 (1950).

[3] In Notes I and II we failed to distinguish between $K(\Pi, 1)$ and $K(\Pi, 1)$ with the zero cell omitted.

[4] In Note II (p. 659) in defining a reduction $f:L \to L'$, we needlessly required that f map L onto L'.

[5] We take this opportunity to correct the following computational error, called to our attention by H. Cartan, in Note II. On p. 662, instead of $A_{11}(J) = (2) + (2) + (2)$, read $A_{11}(J) = (2) + (2)$.

[6] Serre, J. P., *Compt. rend.*, **232**, 142–144 (1951).