GENERALIZED RIEMANN SPACES

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A space of coordinates $x^i(i = 1, \ldots, n)$ with which is associated a non-symmetric tensor g_{ij} is termed a generalized Riemann space. The components g_{ij} and $g'_{\alpha\beta}$ in two coordinate systems x^i and x'^{α} are related by the equation

$$g'_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x}{\partial x'^{\beta}}. \tag{1}$$

The summation convention respecting dummy indices, such as i and j, is employed here and throughout this paper.

Differentiating equation (1) with respect to x'^{γ} we have

$$\frac{\partial g'_{\alpha\beta}}{\partial x'^{\gamma}} = \frac{\partial g_{ij}}{\partial x^k} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}} \frac{\partial x^k}{\partial x'^{\gamma}} + g_{ij} \left(\frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial^2 x^j}{\partial x'^{\beta} \partial x'^{\gamma}} + \frac{\partial^2 x^i}{\partial x'^{\alpha} \partial x'^{\gamma}} \frac{\partial x^j}{\partial x'^{\beta}} \right). \tag{2}$$

The first of the following equations is obtained by interchanging the indices α and γ throughout and the dummy indices i and k in the first term of the right-hand member; the second by interchanging the indices β and γ and the dummy indices j and k:

$$\left.\begin{aligned} \frac{\partial g'_{\gamma\beta}}{\partial x'^{\alpha}} &= \frac{\partial g_{kj}}{\partial x^i} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}} \frac{\partial x^k}{\partial x'^{\gamma}} + g_{ij} \left(\frac{\partial x^i}{\partial x'^{\gamma}} \frac{\partial^2 x^j}{\partial x'^{\beta} \partial x'^{\alpha}} + \frac{\partial^2 x^i}{\partial x'^{\alpha} \partial x'^{\gamma}} \frac{\partial x^j}{\partial x'^{\beta}} \right), \\ \frac{\partial g'_{\alpha\gamma}}{\partial x'^{\beta}} &= \frac{\partial g_{ik}}{\partial x^j} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}} \frac{\partial x^k}{\partial x'^{\gamma}} + g_{ij} \left(\frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial^2 x^j}{\partial x'^{\beta} \partial x'^{\gamma}} + \frac{\partial^2 x^i}{\partial x'^{\alpha} \partial x'^{\beta}} \frac{\partial x^j}{\partial x'^{\gamma}} \right). \end{aligned}\right\} \tag{3}$$

We introduce the notation

$$\Delta_{ijk} = \frac{1}{2} (g_{ik,\,j} + g_{kj,\,i} - g_{ij,\,k}), \tag{4}$$

where throughout this paper a quantity followed by a comma and index indicates the derivative of the quantity with respect to the coordinate with that index; thus $g_{ik,\,j} \equiv \frac{\partial g_{ik}}{\partial x^j}$. By means of this notation one-half the sum of equations (3) minus one-half of equation (2) may be written

$$\Delta'_{\alpha\beta\gamma} = \Delta_{ijk} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}} \frac{\partial x^k}{\partial x'^{\gamma}} + g_{\underline{ij}} \frac{\partial x^i}{\partial x'^{\gamma}} \frac{\partial^2 x^j}{\partial x'^{\alpha} \partial x'^{\beta}} \tag{5}$$

where $g_{\underline{ij}}$ denotes the symmetric part of g_{ij}.

We specify that the determinant of $g_{\underline{ij}}$ is different from zero, and define quantities $g^{\underline{ij}}$ by the equation

$$g_{\underline{ij}}g^{\underline{ik}} = \delta_j^k,$$

where δ_j^k are the Kronecker deltas. By means of the notation

$$\Delta_{ij}{}^h = g^{\underline{hk}}\Delta_{ijk}, \qquad \Delta_{ijk} = g_{\underline{hk}}\Delta_{ij}{}^h, \tag{6}$$

equations (5) are equivalent to[1]

$$\frac{\partial^2 x^h}{\partial x'^\alpha \partial x'^\beta} + \Delta_{ij}{}^h \frac{\partial x^i}{\partial x'^\alpha}\frac{\partial x}{\partial x'^\beta} = \Delta'_{\alpha\beta}{}^\gamma \frac{\partial x^h}{\partial x'^\gamma} \tag{7}$$

From (4) one obtains

$$\Delta_{ikj} + \Delta_{jki} = g_{\underline{ij},\, k}$$

which by (6) may be written

$$g_{\underline{ij},\, k} = g_{\underline{hj}}\Delta_{ik}{}^h + g_{\underline{hi}}\Delta_{jk}{}^h. \tag{8}$$

Also

$$\Delta_{kij} + \Delta_{kji} = g_{\underline{ij},\, k},$$

that is,

$$g_{\underline{ij},\, k} = g_{\underline{hj}}\Delta_{ki}{}^h + g_{\underline{hi}}\Delta_{kj}{}^h. \tag{9}$$

One-half the sum of equations (8) and (9) is

$$g_{\underline{ij},\, k} = g_{\underline{ij}}\Delta_{\underline{ik}}{}^h + g_{\underline{hi}}\Delta_{\underline{jk}}{}^h, \tag{10}$$

where $\Delta_{\underline{ik}}{}^h$ denotes the symmetric part of $\Delta_{ik}{}^h$. From (10) it follows that

$$\Delta_{\underline{ij}}{}^h = \left\{ {h \atop ij} \right\}, \tag{11}$$

where $\left\{ {h \atop ij} \right\}$ is the Christoffel symbol of the second kind in the quantities $g_{\underline{ij}}$.[2]

Denoting by $\Delta_{\check{ij}}{}^h$, $\Delta_{\check{ij}k}$ and $g_{\check{ij}}$ the skew-symmetric parts $\Delta_{ij}{}^h$, Δ_{ijk}, and g_{ij} respectively, we have from (4)

$$g_{\underline{hk}}\Delta_{\check{ij}}{}^h = \Delta_{\check{ij}k} = \frac{1}{2}(g_{\check{ik},\, j} + g_{\check{kj},\, i} + g_{\check{ji},\, k}). \tag{12}$$

A consequence of (12) is

$$g_{\underline{hk}}\Delta_{\check{ij}}{}^h = g_{\underline{hi}}\Delta_{\check{jk}}{}^h = g_{\underline{hj}}\Delta_{\check{ki}}{}^h, \tag{13}$$

which is in agreement with the result of subtracting equation (9) from equation (8). Also one has from (6) and (13)

$$\Delta_i \equiv \Delta_{\underset{\vee}{ij}}{}^{j} = g^{\underline{jk}}\Delta_{\underset{\vee}{ijk}} = g^{\underline{jk}}\Delta_{\underset{\vee}{jki}} = 0. \tag{14}$$

The quantities $\Delta_{ij}{}^{h}$ are coefficients of an affine connection of the space. Any other set $\Gamma_{ij}{}^{h}$ in two coordinate systems x^i and x'^{α} are related by the equations

$$\frac{\partial^2 x^h}{\partial x'^{\alpha}\partial x'^{\beta}} + \Gamma_{ij}{}^{h}\frac{\partial x^i}{\partial x'^{\alpha}}\frac{\partial x^j}{\partial x'^{\beta}} = \Gamma'_{\alpha\beta}{}^{\gamma}\frac{\partial x^h}{\partial x'^{\gamma}} \tag{15}$$

analogous to equations (7). Subtracting this equation from (7) we have

$$(\Gamma_{ij}{}^{h} - \Delta_{ij}{}^{h})\frac{\partial x^i}{\partial x'^{\alpha}}\frac{\partial x^j}{\partial x'^{\beta}} = (\Gamma'_{\alpha\beta}{}^{\gamma} - \Delta'_{\alpha\beta}{}^{\gamma})\frac{\partial x^h}{\partial x^{\gamma}}$$

that is, $a_{ij}{}^{h}$, defined by

$$\Gamma_{ij}{}^{h} - \Delta_{ij}{}^{h} = a_{ij}{}^{h}, \tag{16}$$

are components of a tensor, and for any tensor $a_{ij}{}^{h}$ and $\Delta_{ij}{}^{h}$ one obtains an affine connection.

Interchanging α and β in (15) and the dummy indices i and j, and subtracting the resulting equation from (15), one obtains

$$2\Gamma_{\underset{\vee}{ij}}{}^{h}\frac{\partial x^i}{\partial x'^{\alpha}}\frac{\partial x^j}{\partial x'^{\beta}} = 2\Gamma'_{\underset{\vee}{\alpha\beta}}{}^{\gamma}\frac{\partial x^h}{\partial x'^{\gamma}}$$

that is, the skew-symmetric part $\Gamma_{\underset{\vee}{ij}}{}^{h}$ of $\Gamma_{ij}{}^{h}$ is a tensor. Similarly $\Delta_{\underset{\vee}{ij}}{}^{h}$ is a tensor and by (14) Δ_i is a zero vector.

We write

$$g_{ij,\,k} = g_{hj}\cdot\Delta_{ik}{}^{h} + g_{ih}\Delta_{jk}{}^{h} + a_{ijk} \tag{17}$$

where a_{ijk} are the components of a tensor whose form is to be determined. From (17) we have

$$g_{\underline{ij},\,k} = g_{\underline{hj}}\cdot\Delta_{ik}{}^{h} + g_{\underline{ih}}\Delta_{jk}{}^{h} + \frac{1}{2}(a_{ijk} + a_{jik}).$$

Comparing this equation with equation (8) we note that a_{ijk} is skew-symmetric in the first two indices. Also from (17) one has

$$g_{\underset{\vee}{ij},\,k} = g_{h\underset{\vee}{j}}\cdot\Delta_{ik}{}^{h} + g_{\underset{\vee}{ih}}\Delta_{jk}{}^{h} + a_{ijk} \tag{18}$$

From equations of this form one obtains

$$\frac{1}{2}(g_{\underset{\vee}{ji},\,k} + g_{\underset{\vee}{kj},\,i} + g_{\underset{\vee}{ik}\,j}) = g_{\underset{\vee}{hi}}\Delta_{\underset{\vee}{jk}}{}^{h} + g_{\underset{\vee}{hj}}\Delta_{\underset{\vee}{ki}}{}^{h} + g_{\underset{\vee}{hk}}\Delta_{\underset{\vee}{ij}}{}^{h} - \frac{1}{2}(a_{ijk} + a_{jki} + a_{kij}). \tag{19}$$

This equation is satisfied by

$$a_{ijk} = 2g_{\underset{\vee}{hk}}\Delta_{\underset{\vee}{ij}}{}^{h} - \frac{2}{3}\,\Delta_{\underset{\vee}{ijk}} \tag{20}$$

in consequence of (12) and (13).

If one adds to the right-hand number of (20) the tensor b_{ijk} skew-symmetric in the first two indices the equation (19) is satisfied provided that

$$b_{ijk} + b_{jki} + b_{kij} = 0.$$

An example of a tensor satisfying this condition is

$$b_{ijk} = (\lambda_{i,\,j} - \lambda_{j,\,i})_{,k}$$

where λ_i is any covariant vector.

The condition of integrability of equations (15) is reducible by means of equations of the form (15) to

$$\Gamma_{ijk}{}^{h}\,\frac{\partial x^{i}}{\partial x'^{\alpha}}\,\frac{\partial x^{j}}{\partial x'^{\beta}}\,\frac{\partial x^{k}}{\partial x'^{\gamma}} = \Gamma'{}_{\alpha\beta}{}^{\delta}{}_{\gamma}\,\frac{\partial x^{h}}{\partial x'^{\delta}}$$

where

$$\Gamma_{ij,}{}^{h}{}_{k} = \Gamma_{ij,\,k}^{h} - \Gamma_{i,k\,j}^{h} + \Gamma_{ij}{}^{l}\Gamma_{lk}{}^{h} - \Gamma_{ik}{}^{l}\Gamma_{lj}{}^{h}. \tag{21}$$

Hence $\Gamma_{ij}{}^{h}{}_{k}$ are components of a tensor.

Denoting by Γ_{ij} the tensor $\Gamma_{ij}{}^{h}{}_{h}$, one has

$$\Gamma_{ij} = \Gamma_{ij}{}^{h}{}_{,h} - \Gamma_{ih}{}^{h}{}_{,j} + \Gamma_{ij}{}^{l}\Gamma_{lh}{}^{h} - \Gamma_{ih}{}^{l}\Gamma_{lj}{}^{h}, \tag{22}$$

The symmetric and skew-symmetric parts of Γ_{ij} are

$$\left.\begin{aligned}
\Gamma_{\underline{ij}} &= \Gamma_{\underline{ij}}{}^{h}{}_{,h} - \frac{1}{2}\,(\Gamma_{ih}{}^{h}{}_{,j} + \Gamma_{jh}{}^{h}{}_{,i}) + \Gamma_{\underline{ij}}{}^{l}\Gamma_{lh}{}^{h} - \Gamma_{\underline{ih}}{}^{l}\Gamma_{\underline{jl}}{}^{h} + \Gamma_{\underset{\vee}{ih}}{}^{l}\Gamma_{\underset{\vee}{jl}}{}^{h},\\
\Gamma_{\underset{\vee}{ij}} &= \Gamma_{\underset{\vee}{ij}}{}^{h}{}_{,h} - \frac{1}{2}\,(\Gamma_{ih}{}^{h}{}_{,j} - \Gamma_{jh}{}^{h}{}_{,i}) + \Gamma_{\underset{\vee}{ij}}{}^{h}\Gamma_{lh}{}^{h} - \Gamma_{\underline{ih}}{}^{l}\Gamma_{\underset{\vee}{lj}}{}^{h} - \Gamma_{\underset{\vee}{ih}}{}^{l}\Gamma_{\underline{jl}}{}^{h}.
\end{aligned}\right\} \tag{23}$$

For the connection $\Delta_{ij}{}^{h}$ we have in accordance with (11) and (14)

$$\Gamma_{\underline{ij}}{}^{h} = \left\{ {h \atop ij} \right\}, \qquad \Gamma_{\underset{\vee}{ij}}{}^{h} = \Delta_{\underset{\vee}{ij}}{}^{h}, \qquad \Gamma_{\underset{\vee}{ih}}{}^{h} = \Delta_{i} = 0 \tag{24}$$

Substituting in (23) and noting that[3]

$$\Gamma_{\underline{ih}}{}^{h} = \left\{ {h \atop ih} \right\} = \frac{1}{2}\,\frac{\partial \log g}{\partial x^{i}} \tag{25}$$

where g is the determinant of $g_{\underline{ij}}$, one obtains

$$\Delta_{ij} = -R_{ij} + \Delta_{\underset{\vee}{ih}}{}^{l}\,\Delta_{\underset{\vee}{jl}}{}^{h}, \qquad \Delta_{\underset{\vee}{ij}} = \Delta_{\underset{\vee}{ij}}{}^{h}{}_{/h}, \tag{26}$$

where R_{ij} is the Ricci tensor[4] for the tensor $g_{\underline{ij}}$ and $\Delta_{\underset{\vee}{ij}}{}^h/_h$ is the covariant derivative[5] of $\Delta_{\underset{\vee}{ij}}{}^h$ for the Christoffel symbols $\left\{ {i \atop jk} \right\}$.

From (21) one has

$$\Delta_{hj}{}^h{}_k = \Delta_{hj}{}^h{}_{,k} - \Delta^h_{hk},{}_j = \left\{ {h \atop hj} \right\}_{,k} - \left\{ {h \atop hk} \right\}_{,j} = 0, \tag{27}$$

in consequence of (25).

If the generalized Riemann space is one for which $\Delta_{ij} = 0$, it follows from (26) that

$$R_{ij} = \Delta_{\underset{\vee}{ih}}{}^l \Delta_{\underset{\vee}{jl}}{}^h, \qquad \Delta_{\underset{\vee}{ij}}{}^h{}_{/h} = g^{\underline{hk}} \Delta_{\underset{\vee}{ijk}/h} = 0. \tag{28}$$

The first of these equations involves linear terms in the second derivatives of the quantities $g_{\underline{ij}}$. When the second is written in the second form it is seen from (12) that this equation involves linear terms in the second derivatives of $g_{\underset{\vee}{ij}}$. There are other such equations in view of the fact that the right-hand numbers of the first of equations (28) must satisfy the identities[6]

$$R^h{}_{j/h} = \frac{1}{2} R_{ij},$$

where

$$R^h{}_j = g^{\underline{hi}} R_{ij}, \qquad R = g^{\underline{ij}} R_{ij},$$

and $R^h{}_{j/h}$ is the covariant derivative of $R^h{}_j$ for the symbols $\left\{ {i \atop jk} \right\}$.

1. *R. G.*, p. 19. This is a reference to the author's *Riemannian Geometry*, Princeton University Press.
2. *R. G.*, p. 17.
3. *R. G.*, p. 18.
4. *R. G.*, p. 21.
5. *R. G.*, p. 28.
6. *R. G.*, p. 82.

WEIERSTRASS TRANSFORMS OF POSITIVE FUNCTIONS

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By the Weierstrass transform[1] we mean

$$f(x) = c \int_{-\infty}^{\infty} e^{-(x-y)^2/4} \varphi(y) dy, \qquad c = (4\pi)^{-1/2}. \tag{1}$$

This transform is of particular interest in the general theory of convolution transforms