

GENERALIZED RIEMANN SPACES

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A space of coordinates  $x^i (i = 1, \dots, n)$  with which is associated a non-symmetric tensor  $g_{ij}$  is termed a generalized Riemann space. The components  $g_{ij}$  and  $g'_{\alpha\beta}$  in two coordinate systems  $x^i$  and  $x'^\alpha$  are related by the equation

$$g'_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} \tag{1}$$

The summation convention respecting dummy indices, such as  $i$  and  $j$ , is employed here and throughout this paper.

Differentiating equation (1) with respect to  $x'^\gamma$  we have

$$\frac{\partial g'_{\alpha\beta}}{\partial x'^\gamma} = \frac{\partial g_{ij}}{\partial x^k} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} \frac{\partial x^k}{\partial x'^\gamma} + g_{ij} \left( \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial^2 x^j}{\partial x'^\beta \partial x'^\gamma} + \frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\gamma} \frac{\partial x^j}{\partial x'^\beta} \right) \tag{2}$$

The first of the following equations is obtained by interchanging the indices  $\alpha$  and  $\gamma$  throughout and the dummy indices  $i$  and  $k$  in the first term of the right-hand member; the second by interchanging the indices  $\beta$  and  $\gamma$  and the dummy indices  $j$  and  $k$ :

$$\left. \begin{aligned} \frac{\partial g'_{\gamma\beta}}{\partial x'^\alpha} &= \frac{\partial g_{kj}}{\partial x^i} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} \frac{\partial x^k}{\partial x'^\gamma} + g_{ij} \left( \frac{\partial x^i}{\partial x'^\gamma} \frac{\partial^2 x^j}{\partial x'^\beta \partial x'^\alpha} + \frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\gamma} \frac{\partial x^j}{\partial x'^\beta} \right), \\ \frac{\partial g'_{\alpha\gamma}}{\partial x'^\beta} &= \frac{\partial g_{ik}}{\partial x^j} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} \frac{\partial x^k}{\partial x'^\gamma} + g_{ij} \left( \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial^2 x^j}{\partial x'^\beta \partial x'^\gamma} + \frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\beta} \frac{\partial x^j}{\partial x'^\gamma} \right). \end{aligned} \right\} \tag{3}$$

We introduce the notation

$$\Delta_{ijk} = \frac{1}{2} (g_{ik, j} + g_{kj, i} - g_{ij, k}), \tag{4}$$

where throughout this paper a quantity followed by a comma and index indicates the derivative of the quantity with respect to the coordinate with that index; thus  $g_{ik, j} \equiv \frac{\partial g_{ik}}{\partial x^j}$ . By means of this notation one-half the sum of equations (3) minus one-half of equation (2) may be written

$$\Delta'_{\alpha\beta\gamma} = \Delta_{ijk} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} \frac{\partial x^k}{\partial x'^\gamma} + g_{ij} \frac{\partial x^i}{\partial x'^\gamma} \frac{\partial^2 x^j}{\partial x'^\alpha \partial x'^\beta} \tag{5}$$

where  $g_{ij}$  denotes the symmetric part of  $g_{ij}$ .

We specify that the determinant of  $g_{ij}$  is different from zero, and define quantities  $g^{ij}$  by the equation

$$g_{ij}g^{ik} = \delta_j^k,$$

where  $\delta_j^k$  are the Kronecker deltas. By means of the notation

$$\Delta_{ij}^h = g^{hk}\Delta_{ijk}, \quad \Delta_{ijk} = g_{hk}\Delta_{ij}^h, \quad (6)$$

equations (5) are equivalent to<sup>1</sup>

$$\frac{\partial^2 x^h}{\partial x'^\alpha \partial x'^\beta} + \Delta_{ij}^h \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} = \Delta'_{\alpha\beta}{}^\gamma \frac{\partial x^h}{\partial x'^\gamma} \quad (7)$$

From (4) one obtains

$$\Delta_{ikj} + \Delta_{jki} = g_{ij, k}$$

which by (6) may be written

$$g_{ij, k} = g_{hj}\Delta_{ik}^h + g_{hi}\Delta_{jk}^h. \quad (8)$$

Also

$$\Delta_{kij} + \Delta_{kji} = g_{ij, k}$$

that is,

$$g_{ij, k} = g_{hj}\Delta_{ki}^h + g_{hi}\Delta_{kj}^h. \quad (9)$$

One-half the sum of equations (8) and (9) is

$$g_{ij, k} = g_{ij}\Delta_{ik}^h + g_{hi}\Delta_{jk}^h, \quad (10)$$

where  $\Delta_{ik}^h$  denotes the symmetric part of  $\Delta_{ik}^h$ . From (10) it follows that

$$\Delta_{ij}^h = \{^h_{ij}\}, \quad (11)$$

where  $\{^h_{ij}\}$  is the Christoffel symbol of the second kind in the quantities  $g_{ij}$ .<sup>2</sup>

Denoting by  $\Delta_{ij}^h$ ,  $\Delta_{ijk}$  and  $g_{ij}$  the skew-symmetric parts  $\Delta_{ij}^h$ ,  $\Delta_{ijk}$ , and  $g_{ij}$  respectively, we have from (4)

$$g_{hk}\Delta_{ij}^h = \Delta_{ijk} = \frac{1}{2}(g_{ik, j} + g_{kj, i} + g_{ji, k}). \quad (12)$$

A consequence of (12) is

$$g_{hk}\Delta_{ij}^h = g_{hi}\Delta_{jk}^h = g_{hj}\Delta_{ki}^h, \quad (13)$$

which is in agreement with the result of subtracting equation (9) from equation (8). Also one has from (6) and (13)

$$\Delta_i \equiv \Delta_{ij}^j = g^{jk} \Delta_{ijk} = g^{jk} \Delta_{jki} = 0. \tag{14}$$

The quantities  $\Delta_{ij}^h$  are coefficients of an affine connection of the space. Any other set  $\Gamma_{ij}^h$  in two coordinate systems  $x^i$  and  $x'^\alpha$  are related by the equations

$$\frac{\partial^2 x^h}{\partial x'^\alpha \partial x'^\beta} + \Gamma_{ij}^h \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} = \Gamma'_{\alpha\beta}{}^\gamma \frac{\partial x^h}{\partial x'^\gamma} \tag{15}$$

analogous to equations (7). Subtracting this equation from (7) we have

$$(\Gamma_{ij}^h - \Delta_{ij}^h) \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} = (\Gamma'_{\alpha\beta}{}^\gamma - \Delta'_{\alpha\beta}{}^\gamma) \frac{\partial x^h}{\partial x'^\gamma}$$

that is,  $a_{ij}^h$ , defined by

$$\Gamma_{ij}^h - \Delta_{ij}^h = a_{ij}^h, \tag{16}$$

are components of a tensor, and for any tensor  $a_{ij}^h$  and  $\Delta_{ij}^h$  one obtains an affine connection.

Interchanging  $\alpha$  and  $\beta$  in (15) and the dummy indices  $i$  and  $j$ , and subtracting the resulting equation from (15), one obtains

$$2\Gamma_{ij}^h \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} = 2\Gamma'_{\alpha\beta}{}^\gamma \frac{\partial x^h}{\partial x'^\gamma}$$

that is, the skew-symmetric part  $\Gamma_{ij}^h$  of  $\Gamma_{ij}^h$  is a tensor. Similarly  $\Delta_{ij}^h$  is a tensor and by (14)  $\Delta_i$  is a zero vector.

We write

$$g_{ij, k} = g_{hj} \Delta_{ik}^h + g_{ih} \Delta_{jk}^h + a_{ijk} \tag{17}$$

where  $a_{ijk}$  are the components of a tensor whose form is to be determined. From (17) we have

$$g_{ij, k} = g_{hj} \Delta_{ik}^h + g_{ih} \Delta_{jk}^h + \frac{1}{2} (a_{ijk} + a_{jik}).$$

Comparing this equation with equation (8) we note that  $a_{ijk}$  is skew-symmetric in the first two indices. Also from (17) one has

$$g_{ij, k} = g_{hj} \Delta_{ik}^h + g_{ih} \Delta_{jk}^h + a_{ijk} \tag{18}$$

From equations of this form one obtains

$$\begin{aligned} \frac{1}{2} (g_{ij, k} + g_{kj, i} + g_{ik, j}) &= g_{hi} \Delta_{jk}^h + g_{hj} \Delta_{ki}^h + g_{hk} \Delta_{ij}^h - \\ &\frac{1}{2} (a_{ijk} + a_{jki} + a_{kij}). \end{aligned} \tag{19}$$

This equation is satisfied by

$$a_{ijk} = 2g_{hk}\Delta_{ij}^h - \frac{2}{3}\Delta_{ijk} \tag{20}$$

in consequence of (12) and (13).

If one adds to the right-hand number of (20) the tensor  $b_{ijk}$  skew-symmetric in the first two indices the equation (19) is satisfied provided that

$$b_{ijk} + b_{jki} + b_{kij} = 0.$$

An example of a tensor satisfying this condition is

$$b_{ijk} = (\lambda_i, j - \lambda_j, i), k$$

where  $\lambda_i$  is any covariant vector.

The condition of integrability of equations (15) is reducible by means of equations of the form (15) to

$$\Gamma_{ijk}^h \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} \frac{\partial x^k}{\partial x'^\gamma} = \Gamma'_{\alpha\beta\gamma} \frac{\partial x^h}{\partial x'^\delta}$$

where

$$\Gamma_{ij, k}^h = \Gamma_{ij, k}^h - \Gamma_{i, k, j}^h + \Gamma_{ij}^l \Gamma_{lk}^h - \Gamma_{ik}^l \Gamma_{lj}^h. \tag{21}$$

Hence  $\Gamma_{ij, k}^h$  are components of a tensor.

Denoting by  $\Gamma_{ij}$  the tensor  $\Gamma_{ij, h}^h$ , one has

$$\Gamma_{ij} = \Gamma_{ij, h}^h - \Gamma_{ih, j}^h + \Gamma_{ij}^l \Gamma_{lh}^h - \Gamma_{ih}^l \Gamma_{lj}^h, \tag{22}$$

The symmetric and skew-symmetric parts of  $\Gamma_{ij}$  are

$$\left. \begin{aligned} \Gamma_{ij} &= \Gamma_{ij, h}^h - \frac{1}{2}(\Gamma_{ih, j}^h + \Gamma_{jh, i}^h) + \Gamma_{ij}^l \Gamma_{lh}^h - \Gamma_{ih}^l \Gamma_{jl}^h + \Gamma_{ih}^l \Gamma_{jl}^h, \\ \Gamma_{ij} &= \Gamma_{ij, h}^h - \frac{1}{2}(\Gamma_{ih, j}^h - \Gamma_{jh, i}^h) + \Gamma_{ij}^h \Gamma_{lh}^h - \Gamma_{ih}^l \Gamma_{lj}^h - \Gamma_{ih}^l \Gamma_{jl}^h. \end{aligned} \right\} \tag{23}$$

For the connection  $\Delta_{ij}^h$  we have in accordance with (11) and (14)

$$\Gamma_{ij}^h = \{ \begin{smallmatrix} h \\ ij \end{smallmatrix} \}, \quad \Gamma_{ij}^h = \Delta_{ij}^h, \quad \Gamma_{ik}^h = \Delta_i = 0 \tag{24}$$

Substituting in (23) and noting that<sup>3</sup>

$$\Gamma_{ih}^h = \{ \begin{smallmatrix} h \\ ih \end{smallmatrix} \} = \frac{1}{2} \frac{\partial \log g}{\partial x^i} \tag{25}$$

where  $g$  is the determinant of  $g_{ij}$ , one obtains

$$\Delta_{ij} = -R_{ij} + \Delta_{ih}^l \Delta_{jl}^h, \quad \Delta_{ij} = \Delta_{ij}^h / h, \tag{26}$$

where  $R_{ij}$  is the Ricci tensor<sup>4</sup> for the tensor  $g_{ij}$  and  $\Delta_{ij}^h/h$  is the covariant derivative<sup>5</sup> of  $\Delta_{ij}^h$  for the Christoffel symbols  $\{^i_{jk}\}$ .

From (21) one has

$$\Delta_{hj}^h = \Delta_{hj}^h{}_{,k} - \Delta_{hk}^h{}_{,j} = \{^h_{hj}\}_{,k} - \{^h_{hk}\}_{,j} = 0, \tag{27}$$

in consequence of (25).

If the generalized Riemann space is one for which  $\Delta_{ij} = 0$ , it follows from (26) that

$$R_{ij} = \Delta_{ih}^i \Delta_{j}^h, \quad \Delta_{ij}^h/h = g^{hk} \Delta_{ijk}/h = 0. \tag{28}$$

The first of these equations involves linear terms in the second derivatives of the quantities  $g_{ij}$ . When the second is written in the second form it is seen from (12) that this equation involves linear terms in the second derivatives of  $g_{ij}$ . There are other such equations in view of the fact that the right-hand numbers of the first of equations (28) must satisfy the identities<sup>6</sup>

$$R^h_{j/h} = \frac{1}{2} R_{ij},$$

where

$$R^h_j = g^{hi} R_{ij}, \quad R = g^{ij} R_{ij},$$

and  $R^h_{j/h}$  is the covariant derivative of  $R^h_j$  for the symbols  $\{^i_{jk}\}$ .

1. *R. G.*, p. 19. This is a reference to the author's *Riemannian Geometry*, Princeton University Press.

2. *R. G.*, p. 17.

3. *R. G.*, p. 18.

4. *R. G.*, p. 21.

5. *R. G.*, p. 28.

6. *R. G.*, p. 82.

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## WEIERSTRASS TRANSFORMS OF POSITIVE FUNCTIONS

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By the Weierstrass transform<sup>1</sup> we mean

$$f(x) = c \int_{-\infty}^{\infty} e^{-(x-y)^2/4} \varphi(y) dy, \quad c = (4\pi)^{-1/2}. \tag{1}$$

This transform is of particular interest in the general theory of convolution transforms