and at least one of k_1, k_3, k_6, k_7 is not divisible by p—say k_6. We write

$$A_1^{k_1}A_3^{k_3}A_6^{k_6}A_7^{k_7} = E, \tag{9}$$

and have $E^{p^r} = 1$, so that the period of E is p^s where $s \leqq r$.

Since p does not divide k_6, we have the existence of two integers x, y such that

$$xk_6 + yp^{m_6} = 1. \tag{10}$$

Raise (9) to the x power, and

$$A_6^{p^{m_6}} = 1 \tag{11}$$

to the y power, then multiply; we find, with attention to (10),

$$A_1^{k_1x}A_3^{k_3x}A_6A_7^{k_7x} = E^x; \tag{12}$$

therefore

$$A_6 = A_1^{-k_1x}A_3^{-k_3x}A_7^{-k_7x}E^x. \tag{13}$$

This expresses A_6 as a power-product of other unbarred A's together with an element E whose period $\leqq p^r$. But p^r divides h_6, which by (7) is less than p^{m_6}; therefore $p^r < p^{m_6}$. Hence the period of E is less than that of A_6.

Consequently, also E, as well as A_1, A_3, A_7, was unbarred at the time we came to A_6 in our regular procedure—*this precisely because of our agreement to arrange the elements of G in non-ascending order of the periods.* But then A_6 should be barred, according to our rule of barring and leaving unbarred.

This contradiction establishes the independence of S, and so brings us to our goal of proving the theorem which is the subject of this note.

REPRESENTATIONS OF SEMISIMPLE LIE GROUPS. II

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The object of this note is to announce some further results on representations of a connected semisimple Lie group on a Hilbert space. We shall omit all proofs and assume that the reader is familiar with the contents of an earlier note[1] (quoted henceforward as *RI*).

Let R and C denote the fields of real and complex numbers, respectively, and let G be a connected, simply connected, semisimple Lie group and $\mathfrak{g}_0$ its Lie algebra over R. Let $x \rightarrow Ad(x)(x \in G)$ denote the adjoint representation of G and let K be the complete inverse image in G of some maximal compact subgroup of $Ad(G)$. Then K is a closed connected subgroup of G. Let $\mathfrak{K}_0$ be the Lie algebra of K and let $X \to adX(X \in \mathfrak{g}_0)$ denote the adjoint representation of $\mathfrak{g}_0$. Put $B(X, Y) = sp(adXadY)(X, Y \in \mathfrak{g}_0)$. Let $\mathfrak{P}_0$ be the set of all elements $Y \in \mathfrak{g}_0$ such that $B(X, Y) = 0$ for all $X \in \mathfrak{K}_0$. Then $\mathfrak{g}_0 = \mathfrak{K}_0 + \mathfrak{P}_0$, $\mathfrak{K}_0 \cap \mathfrak{P}_0 = \{0\}$ and there exists an automorphism θ of $\mathfrak{g}_0$ over R such that $\theta(X + Y) = X - Y$ for $X \in \mathfrak{K}_0$, $Y \in \mathfrak{P}_0$. Let $\mathfrak{h}_{\mathfrak{P}_0}$ be a maximal abelian subspace of $\mathfrak{P}_0$. We extend $\mathfrak{h}_{\mathfrak{P}_0}$ to a maximal abelian subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$. Then $\mathfrak{h}_0 = \mathfrak{h}_{\mathfrak{K}_0} + \mathfrak{h}_{\mathfrak{P}_0}$ where $\mathfrak{h}_{\mathfrak{K}_0} = \mathfrak{h}_0 \cap \mathfrak{K}_0$. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$ and let $\mathfrak{P}$, $\mathfrak{K}$, $\mathfrak{h}$, $\mathfrak{h}_{\mathfrak{P}}$, $\mathfrak{h}_{\mathfrak{K}}$ be the subspaces of $\mathfrak{g}$ spanned by $\mathfrak{P}_0$, $\mathfrak{K}_0$, $\mathfrak{h}_0$, $\mathfrak{h}_{\mathfrak{P}_0}$, $\mathfrak{h}_{\mathfrak{K}_0}$ respectively, over C. We extend the bilinear form B and the automorphism θ on $\mathfrak{g}$ by linearity over C. Choose bases $(H_1, \ldots, H_p)$ and $(H_{p+1}, \ldots, H_l)$, respectively, for $\mathfrak{h}_{\mathfrak{P}_0}$ and $\sqrt{-1}\ \mathfrak{h}_{\mathfrak{K}_0}$ over R. Let $\mathfrak{F}$ be the space of all linear functions on $\mathfrak{h}$. Given $\lambda \in \mathfrak{F}$ we can find a unique element $H_\lambda \in \mathfrak{h}$ such that $\lambda(H) = B(H, H_\lambda)$ for all $H \in \mathfrak{h}$. We shall say that λ is real if $H_\lambda \in \mathfrak{h}_{\mathfrak{P}_0} + \sqrt{-1}\ \mathfrak{h}_{\mathfrak{K}_0}$. Moreover if λ is real and $H_\lambda = \sum_{1 \leqslant i \leqslant l} c_i H_i (c_i \in R)$ we say that $\lambda > 0$ if $\lambda \neq 0$ and $c_j > 0$ where j is the least index $(1 \leqslant j \leqslant l)$ such that $c_j \neq 0$. We know that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\theta\mathfrak{h} = \mathfrak{h}$. For every root α of $\mathfrak{g}$ with respect to $\mathfrak{h}$ let $X_\alpha \neq 0$ denote an element in $\mathfrak{g}$ such that $[H, X_\alpha] = \alpha(H)X_\alpha(H \in \mathfrak{h})$. Let P be the set of all roots $\alpha > 0$. For any $\lambda \in \mathfrak{F}$ let $\theta\lambda$ denote the linear function given by $\theta\lambda(H) = \lambda(\theta H)(H \in \mathfrak{h})$. Then if α is α root $\theta\alpha$ is also a root. Let P_+ be the set of all $\alpha \in P$ such that $\theta\alpha < 0$ and P the remaining set of positive roots. Iwasawa[2] has shown that $X_\alpha, X_{-\alpha} \in \mathfrak{K}$ for $\alpha \in P$. Put

$$\mathfrak{N} = \sum_{\alpha \in P_+} CX_\alpha,\ \mathfrak{M} = \mathfrak{h}_{\mathfrak{K}} + \sum_{\alpha \in P} CX_\alpha + \sum_{\alpha \in P} CX_{-\alpha}.$$

Then $\mathfrak{N}$ is a nilpotent subalgebra of $\mathfrak{g}$ and $\mathfrak{M}$ is a subalgebra of $\mathfrak{K}$. Put $\mathfrak{N}_0 = \mathfrak{N} \cap \mathfrak{g}_0$, $\mathfrak{M}_0 = \mathfrak{M} \cap \mathfrak{K}_0$. Then $\mathfrak{g}_0 = \mathfrak{K}_0 + \mathfrak{h}_{\mathfrak{P}_0} + \mathfrak{N}_0$ where the sum is direct[2] and $[\mathfrak{M}, \mathfrak{N}] \subset \mathfrak{N}$, $[\mathfrak{M}, \mathfrak{h}_{\mathfrak{P}}] = \{0\}$. Moreover $\mathfrak{K}$ and $\mathfrak{M}$ are reductive algebras, i.e., they are the direct sums of their centers and their derived algebras which are semisimple. Let $\mathfrak{C}$ be the center of $\mathfrak{K}$. Put $\mathfrak{C}_0 = \mathfrak{C} \cap \mathfrak{g}_0$ and $\mathfrak{K}_0' = [\mathfrak{K}_0, \mathfrak{K}_0]$. Let D, K', A_+, M and N, respectively, be the analytic subgroups of G corresponding to $\mathfrak{C}_0$, $\mathfrak{K}_0'$, $\mathfrak{h}_{\mathfrak{P}_0}$, $\mathfrak{M}_0$ and $\mathfrak{N}_0$. Then M is closed and the mappings $(\gamma, u) \to \gamma u(\gamma \in D, u \in K')$ and $(v, h, n) \to vhn(v \in K, h \in A_+, n \in N)$ are analytic isomorphisms of DXK' with K and $K \times A_+ \times N$ with G, respectively.

Let Ω, Ω' and ω, respectively, be the set of all equivalence classes of finite-dimensional irreducible representations of K, K' and M. Then if $\mathfrak{D} \in \Omega$ and $\sigma \in \mathfrak{D}$ it is easily seen that $\sigma(K')$ is irreducible and $\sigma(M)$ is fully reducible. We denote by $\mathfrak{D}'$ the equivalence class of the irreducible representation of K' defined by σ. Also if $\delta \in \omega$ we say that $\delta < \mathfrak{D}$ if δ occurs in the reduction of $\sigma(M)$. Let $\alpha \in \delta \in \omega$. Then α defines α representation β of $\mathfrak{M}_0$ and therefore of $\mathfrak{M}$. Let λ and μ be any two weights of β with respect to $\mathfrak{h}_{\mathfrak{K}}$. We extend λ and μ on $\mathfrak{h}$ by defining them to be zero on $\mathfrak{h}_{\mathfrak{P}}$. Then it is easily seen that $\lambda - \mu$ is a real linear function on $\mathfrak{h}$. Hence β has a weight λ such that $\lambda - \mu > 0$ for every other weight μ. Clearly λ depends on δ alone and we shall call it the highest weight of δ.

Let $\mathfrak{B}$ be the universal enveloping algebra of $\mathfrak{g}$ and let $\mathfrak{Z}$ be the center of $\mathfrak{B}$. Let $C[x]$ denote the ring of all commutative polynomials in l independent variables $x_1, \ldots, x_l$ with coefficients in C. We denote by β the isomorphic mapping of $C[x]$ into $\mathfrak{B}$ given by $\beta(x_1^{m_1}x_2^{m_2}\ldots x_l^{m_l}) = H_1^{m_1}\ldots H_l^{m_l}$. For any $z \in \mathfrak{Z}$ there exists a unique element[3] $\chi_x(z) \in C[x]$ such that $z - \beta(\chi_x(z)) \in \sum_{\alpha \in P} \mathfrak{B}X_\alpha$. Λ being any linear function on $\mathfrak{h}$ we denote by $\chi_\Lambda(z)$ the value of the polynomial $\chi_x(z)$ at $x_i = \Lambda(H_i)| \leqslant i \leqslant l$. Then the mapping $\chi_\Lambda: z \rightarrow \chi_\Lambda(z) (z \in \mathfrak{Z})$ is a homomorphism of $\mathfrak{Z}$ into C and conversely every homomorphism of $\mathfrak{Z}$ into C is of the form[3] χ_Λ for some $\Lambda \in \mathfrak{F}$. Let $\rho = {}^1/_2 \sum_{\alpha \in P} \alpha$ and let W be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then if $\Lambda_1, \Lambda_2 \in \mathfrak{F}$, $\chi_{\Lambda_1} = \chi_{\Lambda_2}$ if and only if[3] $s(\Lambda_1 + \rho) = \Lambda_2 + \rho$ for some $s \in W$.

We denote by Ω_F the set of all $\mathfrak{D} \in \Omega$ for which there exists a finite-dimensional representation π of G such that $\mathfrak{D}'$ occurs in the reduction of $\pi(K')$.

THEOREM 1. *Let π be a quasisimple[1] representation of G on a Banach space with the infinitesimal character[4] χ. Suppose $\mathfrak{D}$ is an element in Ω_F such that $\mathfrak{D}$ occurs[5] in π. Then there exists a linear function Λ on $\mathfrak{h}$ and a $\delta \in \omega$ such that $\delta < \mathfrak{D}$ and $\chi = \chi_\Lambda$ and $\Lambda(H) = \lambda_\delta(H) (H \in \mathfrak{h}_{\mathfrak{K}})$, where λ_δ is the highest weight of δ. Conversely suppose we are given a linear function Λ on $\mathfrak{h}$ and $\mathfrak{D} \in \Omega$ such that Λ coincides on $\mathfrak{h}_{\mathfrak{K}}$ with the highest weight λ_δ of some $\delta < \mathfrak{D}$ $(\delta \in \omega)$. Then there exists a quasisimple irreducible representation π of G on a Hilbert space with the infinitesimal character χ_Λ such that $\mathfrak{D}$ occurs in π.*

Remarks.—It seems likely that the first part of the above theorem is actually true for all $\mathfrak{D} \in \Omega$ and not merely for $\mathfrak{D} \in \Omega_F$, but so far it has not been possible to prove this. Notice that if G is a complex semisimple group $\Omega = \Omega_F$ and so in this case the theorem holds without any restriction on $\mathfrak{D}$.

Let π_1 and π_2 be two quasisimple representations of G on the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. For any $\mathfrak{D} \in \Omega$ let $\mathfrak{H}_{i,\mathfrak{D}}$ denote the set of all elements in $\mathfrak{H}_i$ which transform[1] under $\pi_i(K)$ according to $\mathfrak{D} (i = 1, 2)$. Put $\mathfrak{H}_i^0 = \sum_{\mathfrak{D} \in \Omega} \mathfrak{H}_{i,\mathfrak{D}}$. Then we get a representation π_i^0 of $\mathfrak{B}$ on $\mathfrak{H}_i^0$ such that

$$\pi_i^0(X)\psi = \lim_{t \rightarrow 0} \frac{1}{t} \{\pi_i(\exp t X)\psi - \psi\} \qquad (\psi \in \mathfrak{H}_i^0, X \in \mathfrak{g}_0, t \in R).$$

We say that π_1 and π_2 are infinitesimally equivalent if the representations π_1^0 and π_2^0 are algebraically equivalent, i.e., if there exists an isomorphism α of $\mathfrak{H}_1^0$ onto $\mathfrak{H}_2^0$ such that $\pi_2^0(b)\alpha\psi = \alpha\pi_1^0(b)\psi(b \in \mathfrak{B}, \psi \in \mathfrak{H}_1^0)$. Clearly if π_1 and π_2 are equivalent they are also infinitesimally equivalent. Conversely it can be shown that if π_1 and π_2 are both unitary, their infinitesimal equivalence implies their equivalence in the usual sense.

Let π be a quasisimple irreducible representation of G on a Hilbert space. We know (see Theorem 3 of *RI*) that dim $\mathfrak{H}_{\mathfrak{D}} < \infty$ for all $\mathfrak{D} \in \Omega$. Moreover, we may assume without loss of generality that the subspaces $\mathfrak{H}_{\mathfrak{D}}$ are all mutually orthogonal for distinct $\mathfrak{D}$. Let $E_{\mathfrak{D}}$ denote the orthogonal projection on $\mathfrak{H}$ on $\mathfrak{H}_{\mathfrak{D}}$. Put

$$\varphi_{\mathfrak{D}}^{\pi}(x) = sp(E_{\mathfrak{D}}\pi(x)E_{\mathfrak{D}}) \qquad (x \in G).$$

Then we can restate Theorem 7 of *RI* in a slightly improved form as follows.

THEOREM 2. *Let π_1, π_2 be irreducible quasisimple representations of G on two Hilbert spaces. Suppose that for some $\mathfrak{D} \in \Omega$ and $c \in C$, $\varphi_{\mathfrak{D}}^{\pi_1} = c\varphi_{\mathfrak{D}}^{\pi_2} \neq 0$. Then π_1 and π_2 are infinitesimally equivalent. Conversely if π_1 and π_2 are infinitesimally equivalent $\varphi_{\mathfrak{D}}^{\pi_1} = \varphi_{\mathfrak{D}}^{\pi_2}$ for all $\mathfrak{D} \in \Omega$.*

We have seen above that every element $x \in G$ can be written uniquely in the form $x = vhn(v \in K, h \in A_+, n \in N)$. For any $v \in K$ and $x \in G$ let v_x and $H(x, v)$ denote the unique elements in K and $\mathfrak{h}_{\mathfrak{P}_0}$, respectively, such that $xv = v_x(\exp H(x, v))n$ for some $n \in N$. Moreover let $\Gamma(v)$ denote the element in $\mathfrak{C}_0$ such that $v = (\exp \Gamma(v))u$ for some $u \in K'$. Put $\Gamma(x, v) = \Gamma(v_x) - \Gamma(v)$. Let Z be the center of G and let $z \rightarrow z^*$ denote the adjoint representation of G. Then $K \supset Z$ and K^* is compact. It is easily seen that $(va)_x = v_x a$, $H(x, va) = H(x, v)$, $\Gamma(x, va) = \Gamma(x, v)(a \in Z)$. Hence we may write $H(x, v) = H(x, v^*)$, $\Gamma(x, v) = \Gamma(x, v^*)$. Let dv^* denote the Haar measure on K^* such that $\int_{K^*} dv^* = 1$. For any $\mathfrak{D} \in \Omega$ let $\mu_{\mathfrak{D}}$ denote the linear function on $\mathfrak{C}$ such that

$$\sigma(\exp \Gamma) = e^{\mu_{\mathfrak{D}}(\Gamma)}\sigma(1) \qquad (\Gamma \in \mathfrak{C}_0)$$

for $\sigma \in \mathfrak{D}$. Also let $d(\mathfrak{D})$ denote the degree of σ.

THEOREM 3. *Let π be a quasisimple irreducible representation of G on a Hilbert space $\mathfrak{H}$ and let $\mathfrak{D}$ be an element in Ω such that $d(\mathfrak{D}) = 1$ and $\mathfrak{D}$ occurs*[5] *in π. Then dim $\mathfrak{H}_{\mathfrak{D}} = 1$ and there exists a linear function Λ on $\mathfrak{h}$ such that χ_Λ is the infinitesimal*[4] *character of π and*

$$\varphi_{\mathfrak{D}}^{\pi}(x) = \int_{K^*} e^{\mu_{\mathfrak{D}}(\Gamma(x, v^*))}e^{\Lambda(H(x, v^*))}\, dv^* \qquad (x \in G).$$

[1] Harish-Chandra, PROC. NATL. ACAD. SCI., **37**, 170–173 (1951).

[2] Iwasawa, K., *Ann. Math.*, **50**, 507–558 (1949).

[3] Harish-Chandra, *Trans. Am. Math. Soc.*, **70**, 28–96 (1951).

[4] The infinitesimal character of π was called simply the character of π in *RI*.

[5] This means that $\mathfrak{D}$ occurs in the reduction of $\pi(K)$.