APPROXIMATIONS EXCEEDING 1300 DECIMALS FOR $\sqrt{3}$, $1/\sqrt{3}$, SIN $(\pi/3)$ AND DISTRIBUTION OF DIGITS IN THEM

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This note is practically a sequel to an earlier communication[1] relating to $\sqrt{2}$ and $1/\sqrt{2}$. Hence acquaintance with the first paper will be assumed. The present investigation had two goals, namely (a) to extend appreciably our present approximations for $\sqrt{3}$ and $1/\sqrt{3}$, and (b) to make a brief statistical examination of the distribution of digits in these constants.

Without reference to previous values the calculation of $\sqrt{3}$ was begun by the obvious long-hand method and terminated with the 35th decimal figure. This gave 35 antecedent zeros and 36 significant digits for the corresponding value of δ. [See formulae (1), (2), (3) in the first note.] For the third approximation a was assigned 408 digits, i.e., 407 decimal figures. The associated value of δ had 406 ante-zeros and 408 significant figures. $N = 3$. $a\delta/6 = (0.381835\cdots) \times 10^{-407}$, $a\delta^2/24 = (0.126264\cdots) \times 10^{-814}$, $5a\delta^3/432 = (0.463924\cdots) \times 10^{-1222}$. The data produced in the third and final stage of the work were checked with extreme care with auxiliary moduli, and by other devices. Finally the process of cross-multiplication[2] was applied to the first 1314 figures of the dedecimalized value of $\sqrt{3}$ and a continuous sequence of 1312 nines resulted.

The newly computed values for $\sqrt{3}$, $1/\sqrt{3}$ and $(\sqrt{3})/2$ are given in table 1. [The omission of $1/\sqrt{2}$ from the earlier note and the inclusion of $1/\sqrt{3}$ in this place were made intentionally.]

Table 2 presents the values of χ_n^2 and $p_n(= 1 - P_n)$ for the three newly extended constants, and it corresponds to table 2 of the earlier note. Comparison of the fifth columns of these tables brings out at once the striking contrast between the probability values for the reciprocals of $\sqrt{2}$ and $\sqrt{3}$. For $1/\sqrt{2}$ most of the numbers are unusually small while the corresponding data for $1/\sqrt{3}$ are large, especially at $n = 200$ and 300. The conjecture that the small values of p_n for $(\sqrt{2})/2$ may be due to division by 2 is refuted immediately by the steady large values ($n > 600$) in the seventh column of the present table 2. [The asterisks in the columns for χ_n^2 indicate that all subsequent meaningless figures have been omitted.]

Many formulae for statistical probability are so constructed as to involve the squares of the deviations and hence they mask the signs and implications of these differences. In the particular case of $1/\sqrt{2}$ and $1/\sqrt{3}$ the changes of the deviations by excess and deficiency seem to be interesting and perhaps instructive. For this reason table 3 is presented for inspection. The upper line of each double row refers to $1/\sqrt{2}$ and the lower line of the paired rows to $1/\sqrt{3}$. In $1/\sqrt{3}$ the consistently large deficiencies of 4 and the steady growth of 7 may be noted. In the special case of $1/\sqrt{3}$ there are 51 digits between the 922nd and 974th decimal places among which 1 does not occur. Also 4 is absent from the 53 decimal cells bounded by the 150th and 204th places.

It seems appropriate to compare the final approximation to $\sqrt{3}$ made by J. M. Boorman[3] with the accurate value given here. On page 207 of the first volume of the *Mathematical Magazine* he stated that "Below is the corrected root which I believe to be true to at least 300 places of decimals, having made the revised compute with great care." Nevertheless Boorman's datum included 422 decimal figures of which the first 222 are correct and the succeeding 200 digits constitute an error of excess.

TABLE 1

			$\sqrt{3}$ =					±1.7320	50807
56887	72935	27446	34150	58723	66942	80525	38103	80628	05580
69794	51933	01690	88000	37081	14618	67572	48575	67562	61414
15406	70302	99699	45094	99895	24788	11655	51209	43736	48528
09323	19023	05582	06797	48201	01084	67492	32650	15312	34326
69033	22886	65067	22546	68921	83797	12270	47131	66036	78615
88019	04998	65373	79859	38946	76503	47506	57605	07566	18348
12960	61009	47602	18719	03250	83145	82952	39598	32997	78982
45082	88714	46383	29173	47224	16398	45878	55397	66795	80638
18353	66611	08431	73780	89437	83161	02088	30552	49016	70023
52071	11442	88695	99095	63657	97087	16849	80728	99493	29648
42830	20786	40860	39887	38697	53758	23173	17831	39599	29830
07838	70287	70539	13369	56331	21037	07264	01924	91067	68231
19928	83756	41141	42201	67427	52102	37299	42708	31059	89845
94759	87664	28889	77961	47837	95839	02288	54852	90357	60338
52808	06438	19723	44661	05968	97228	72865	26415	38226	64698
42002	11954	84155	27844	11812	86534	50703	51916	50016	68929
44154	80846	07127	71439	99762	92683	46295	77438	36189	51101
27148	63874	69765	45982	45178	85509	75379	01388	06649	61911
96222	29571	10555	24292	37231	92197	73826	25616	31468	84203
28537	16682	93864	96119	17049	73883	63954	95938	14575	76718
53373	63312	59108	99655	42462	48347	87197	60523	59977	69192
32357	02203	05302	84038	59154	14971	07242	95592	06706	20250
95201	75963	18587	27663	59975	28366	34310	80150	66585	37106
47328	53862	59222	60582	22051	04036	80270	29750	47987	28079
46165	81004	17052	68194	00190	95733	46217	59438	93670	24932
04226	91034	36981	24637	20111	18526	10842	68910	29972	03112
02100	06···								
			$1/\sqrt{3}$ =					±0.5773	50269
18962	57645	09148	78050	19574	55647	60175	12701	26876	01860
23264	83977	67230	29333	45693	71539	55857	49525	22520	87138
05135	56767	66566	48364	99965	08262	70551	83736	47912	16176
03107	73007	68527	35599	16067	00361	55830	77550	05104	11442

TABLE 1—*Continued*

23011	07628	88355	74182	22973	94599	04090	15710	55345	59538
62673	01666	21791	26619	79648	92167	82502	19201	69188	72782
70986	87003	15867	39573	01083	61048	60984	13199	44332	59660
81694	29571	48794	43057	82408	05466	15292	85132	55598	60212
72784	55537	02810	57926	96479	27720	34029	43517	49672	23341
17357	03814	29565	33031	87885	99029	05616	60242	99831	09882
80943	40262	13620	13295	79565	84586	07724	39277	13199	76610
02612	90095	90179	71123	18777	07012	35754	67308	30355	89410
39976	27918	80380	47400	55809	17367	45766	47569	43686	63281
98253	29221	42963	25987	15945	98613	00762	84950	96785	86779
50936	02146	06574	48220	35322	99076	24288	42138	46075	54899
47334	03984	94718	42614	70604	28844	83567	83972	16672	22976
48051	60282	02375	90479	99920	97561	15431	92479	45396	50367
09049	54624	89921	81994	15059	61836	58459	67129	35549	87303
98740	76523	70185	08097	45743	97399	24608	75205	43822	94734
42845	72227	64621	65373	05683	24627	87984	98646	04858	58906
17791	21104	19702	99885	14154	16115	95732	53507	86659	23064
10785	67401	01767	61346	19718	04990	35747	65197	35568	73416
98400	58654	39529	09221	19991	76122	11436	93383	55528	45702
15776	17954	19740	86860	74017	01345	60090	09916	82662	42693
15388	60334	72350	89398	00063	65244	48739	19812	97890	08310
68075	63678	12327	08212	40037	06175	36947	56303	43324	01037
34033	35···								

$\sin(\pi/3) =$ 0.8660 25403

78443	86467	63723	17075	29361	83471	40262	69051	90314	02790
34897	25966	50845	44000	18540	57309	33786	24287	83781	30707
07703	35151	49849	72547	49947	62394	05827	75604	71868	24264
04661	59511	52791	03398	74100	50542	33746	16325	07656	17163
34516	61443	32533	61273	34460	91898	56135	23565	83018	39307
94009	52499	32686	89929	69473	38251	73753	28802	53783	09174
06480	30504	73801	09359	51625	41572	91476	19799	16498	89491
22541	44357	23191	64586	73612	08199	22939	27698	83397	90319
09176	83305	54215	86890	44718	91580	51044	15276	24508	35011
76035	55721	44347	99547	81828	98543	58424	90364	49746	64824
21415	10393	20430	19943	69348	76879	11586	58915	69799	64915
03919	35143	85269	56684	78165	60518	53632	00962	45533	84115
59964	41878	20570	71100	83713	76051	18649	71354	15529	94922
97379	93832	14444	88980	73918	97919	51144	27426	45178	80169
26404	03219	09861	72330	52984	48614	36432	63207	69113	32349
21001	05977	42077	63922	05906	43267	25351	75958	25008	34464
72077	40423	03563	85719	99881	46341	73147	88719	18094	75550
63574	31937	34882	72991	22589	42754	87689	50694	03324	80955
98111	14785	55277	62146	18615	96098	86913	12808	15734	42101
64268	58341	46932	48059	58524	86941	81977	47969	07287	88359
26686	81656	29554	49827	71231	24173	93598	80261	79988	84596
16178	51101	52651	42019	29577	07485	53621	47796	03353	10125
47600	87981	59293	63831	79987	64183	17155	40075	33292	68553
23664	26931	29611	30291	11025	52018	40135	14875	23993	64039
73082	90502	08526	34097	00095	47866	73108	79719	46835	12466
02113	45517	18490	62318	60055	59263	05421	34455	14986	01556
01050	03···								

TABLE 2

	$\sqrt{3}$		$1/\sqrt{3}$		$(\sqrt{3})/2$	
n	χ_n^2	p_n	χ_n^2	p_n	χ_n^2	p_n
100	8.6	0.5211	7.8	0.4445	5.8	0.2414
200	6.	0.2603	21.5	0.9891	8.3	0.4956
300	5.6	0.2241	$17.4\dot{6}$	0.9565	$6.2\dot{6}$	0.2876
400	4.45	0.1244	10.95	0.7140	4.1	0.0960
500	8.04	0.4690	11.2	0.7345	6.32	0.2931
600	11.3	0.7427	$13.0\dot{6}$	0.8264	8.3	0.4956
700	11.4	0.7505	11.914*	0.7739	12.428*	0.7973
800	10.275	0.6586	7.275	0.3908	10.05	0.6401
900	$10.6\dot{2}$	0.6871	$9.4\dot{8}$	0.5941	11.6	0.7596
1000	10.5	0.6771	9.78	0.6180	12.64	0.8070
1100	$8.\dot{2}\dot{7}$	0.4928	$10.1\dot{4}\dot{5}$	0.6480	$11.\dot{4}9\dot{0}$	0.7547
1200	$9.4\dot{6}$	0.5922	11.55	0.7573	11.85	0.7710
1300	8.215*	0.4870	9.215*	0.5716	11.276*	0.7408

TABLE 3

n	DIGIT 0	1	2	3	4	5	6	7	8	9	ΣD^2
100	− 1	− 2	− 3	+ 4	+ 1	− 2	+1	− 1	+ 4	− 1	54
	− 1	− 2	0	− 1	− 3	+ 6	+1	+ 4	− 3	− 1	78
200	− 2	− 3	− 2	+ 4	− 3	0	+5	+ 1	0	0	68
	+ 3	− 3	− 4	− 1	−10	+11	+7	+ 8	− 6	− 5	430
300	− 2	− 3	− 3	+ 3	− 1	+ 1	+2	+ 3	− 2	+ 2	54
	+ 3	+ 2	− 1	− 5	−12	+12	+8	+ 6	− 9	− 4	524
400	− 3	− 2	− 4	+ 1	− 2	− 2	+3	+ 9	− 1	+ 1	130
	+ 4	+ 2	− 3	− 6	−12	+11	+7	+ 5	− 5	− 3	438
500	− 4	− 1	− 4	+ 2	− 1	+ 2	−1	+ 5	− 1	+ 3	78
	+ 4	0	+ 1	− 6	−14	+13	+5	+ 7	− 8	− 2	560
600	− 2	− 2	− 1	+ 4	+ 2	− 3	−4	+ 9	0	− 3	144
	+ 7	+ 2	+ 2	− 6	−19	+10	+3	+11	−10	0	784
700	+ 1	0	+ 4	+ 6	+ 2	− 4	−8	+ 7	− 7	− 1	236
	+ 7	− 1	+ 1	− 7	−20	+11	+4	+10	− 9	+ 4	834
800	+ 3	− 1	+ 4	+ 5	+ 4	− 3	−5	+ 3	− 6	− 4	162
	+ 6	− 6	+ 2	− 8	−14	+ 8	+4	+11	− 6	+ 3	582
900	+ 3	− 2	+10	+ 5	+ 6	− 4	−5	− 2	− 8	− 3	292
	+ 5	− 7	+ 3	−12	−14	+ 9	+5	+ 9	−10	+12	854
1000	+ 4	− 4	+ 8	+ 2	+ 9	+ 2	−5	− 4	− 8	− 4	306
	+ 4	−15	+ 5	−12	−10	+ 9	+4	+13	− 9	+11	978
1100	+ 5	− 4	+ 5	+ 3	+11	+ 4	−6	+ 1	− 9	−10	430
	+ 4	− 8	− 1	−16	−11	+11	+4	+16	−11	+12	1116
1200	+ 6	− 3	+ 4	+ 2	+14	+ 2	−6	− 1	− 8	−10	466
	+ 6	− 5	− 4	−19	−12	+13	+4	+15	−13	+15	1386
1300	+ 3	− 4	+ 7	− 3	+19	+ 4	−6	+ 1	− 8	−13	730
	+10	− 8	− 4	−14	−14	+ 9	+6	+14	−12	+13	1198

Beginning with the 223rd place the first fifteen figures of his error are 28867 51345 94813.

[1] These PROCEEDINGS, 37, 63–67 (1951).

[2] Uhler, H. S., "Miscellaneous Hints for and Experiences in Computation," *Scripta Mathematica*, 16, 31–42 (1950).

[3] Boorman, J. Marcus. Square-Root Notes. *The Mathematical Magazine*, 1, 207, 208 (1887).

THE RELATION OF ADRENAL HORMONES TO THE PATHOGENESIS OF EXPERIMENTAL NEPHRITIS

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Although antikidney serum was first studied by Lindemann[1] in 1900, Masugi[2] first produced nephritis in rats by the administration of antikidney serum in 1931. Following Masugi, several other investigators have studied in some detail the pathogenesis of nephritis produced by antikidney serum in the rat, notably Smadel and his associates[3–5] and Heymann.[6] Consideration of the reported results impresses one with the variability of experimental nephritis produced by nephrotoxic sera. The administration of a given dose produced somewhat different diseases in the hands of each investigator, and this circumstance has been variously attributed to the diet, the strain of rat, and other differences in the conditions.

As part of a comprehensive study, we have investigated the variability of experimental nephritis as influenced by a change in one experimental condition, when all others are held constant. In this manner, we have investigated the effects of dose, route of administration, time intervals, and individual preparations. In the course of this investigation, effects were noted that could be correlated closely with certain manifestations of adrenal function, and these data are related here.

Rabbit anti-rat-kidney serum was prepared by immunizing rabbits for a period of 4 months or more with a suspension of rat kidney tissue. This is the same preparation (NTG) which has been described briefly in a previous paper.[7] However, in this experiment the serum was adsorbed with a suspension of sheep erythrocytes to remove Forssman antibodies, but adsorption with rat erythrocytes was omitted. The antibody globulin was precipitated by one-third saturation with ammonium sulfate at pH 7.8. This NTG was found by the Tiselius pattern to be composed entirely of γ-globulin. The minimum serological activity was determined by a precipitin test with the soluble portion of the original antigen, and the