ON THE GREEN'S FUNCTIONS OF QUANTIZED FIELDS. I

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The temporal development of quantized fields, in its particle aspect, is described by propagation functions, or Green's functions. The construction of these functions for coupled fields is usually considered from the viewpoint of perturbation theory. Although the latter may be resorted to for detailed calculations, it is desirable to avoid founding the formal theory of the Green's functions on the restricted basis provided by the assumption of expandability in powers of coupling constants. These notes are a preliminary account of a general theory of Green's functions, in which the defining property is taken to be the representation of the fields of prescribed sources.

We employ a quantum dynamical principle for fields which has been described elsewhere.[1] This principle is a differential characterization of the function that produces a transformation from eigenvalues of a complete set of commuting operators on one space-like surface to eigenvalues of another set on a different surface,[2]

$$\delta(\zeta_1', \sigma_1|\zeta_2'', \sigma_2) = i(\zeta_1', \sigma_1|\delta\int_{\sigma_2}^{\sigma_1}(dx)\mathcal{L}|\zeta_2'', \sigma_2). \tag{1}$$

Here $\mathcal{L}$ is the Lagrange function operator of the system. For the example of coupled Dirac and Maxwell fields, with external sources for each field, the Lagrange function may be taken as

$$\begin{aligned}\mathcal{L} = -{}^1/_4[\bar{\psi}, \gamma_\mu(-i\partial_\mu - eA_\mu)\psi + m\psi] + {}^1/_2[\bar{\psi}, \eta] + \\ \text{Herm. conj.} + {}^1/_4F_{\mu\nu}{}^2 - {}^1/_4\{F_{\mu\nu}, \partial_\mu A_\nu - \partial_\nu A_\mu\} + J_\mu A_\mu,\end{aligned} \tag{2}$$

which implies the equations of motion

$$\begin{gathered}\gamma_\mu(-i\partial_\mu - eA_\mu)\psi + m\psi = \eta, \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \partial_\nu F_{\mu\nu} = J_\mu + j_\mu,\end{gathered} \tag{3}$$

where

$$j_\mu = e{}^1/_2[\bar{\psi}, \gamma_\mu\psi]. \tag{4}$$

With regard to commutation relations, we need only note the anticommutativity of the source spinors with the Dirac field components.

We shall restrict our attention to changes in the transformation function that arise from variations of the external sources. In terms of the notation

$$\begin{gathered}(\zeta_1', \sigma_1|\zeta_2'', \sigma_2) = \exp i\mathcal{W}, \\ (\zeta_1', \sigma_1|F(x)|\zeta_2'', \sigma_2)/(\zeta_1', \sigma_1|\zeta_2'', \sigma_2) = \langle F(x)\rangle,\end{gathered} \tag{5}$$

the dynamical principle can then be written

$$\delta \mathcal{W} = \int_{\sigma_2}^{\sigma_1} (dx)\langle \delta \mathcal{L}(x)\rangle, \tag{6}$$

where

$$\langle \delta \mathcal{L}(x)\rangle = \langle \bar{\psi}(x)\rangle \delta\eta(x) + \delta\bar{\eta}(x)\langle \psi(x)\rangle + \langle A_\mu(x)\rangle \delta J_\mu(x). \tag{7}$$

The effect of a second, independent variation is described by

$$\delta'\langle \delta \mathcal{L}(x)\rangle = i \int_{\sigma_2}^{\sigma_1} (dx')[\langle(\delta \mathcal{L}(x)\delta' \mathcal{L}(x'))_+\rangle - \langle \delta \mathcal{L}(x)\rangle\langle \delta' \mathcal{L}(x')\rangle], \tag{8}$$

in which the notation $(\ )_+$ indicates temporal ordering of the operators. As examples we have

$$\delta_\eta\langle \psi(x)\rangle = i \int_{\sigma_2}^{\sigma_1} (dx')[\langle(\psi(x)\bar{\psi}(x')\delta\eta(x'))_+\rangle - \langle \psi(x)\rangle\langle \bar{\psi}(x')\delta\eta(x')\rangle], \tag{9}$$

and

$$\delta_J\langle \psi(x)\rangle = i \int_{\sigma_2}^{\sigma_1} (dx')[\langle(\psi(x)A_\mu(x'))_+\rangle - \langle \psi(x)\rangle\langle A_\mu(x')\rangle]\delta J_\mu(x'). \tag{10}$$

The latter result can be expressed in the notation

$$-i(\delta/\delta J_\mu(x'))\langle \psi(x)\rangle = \langle(\psi(x)A_\mu(x'))_+\rangle - \langle \psi(x)\rangle\langle A_\mu(x')\rangle, \tag{11}$$

although one may supplement the right side with an arbitrary gradient. This consequence of the charge conservation condition, $\partial_\mu J_\mu = 0$, corresponds to the gauge invariance of the theory.

A Green's function for the Dirac field, in the absence of an actual spinor source, is defined by

$$\delta_\eta\langle \psi(x)\rangle]_{\eta=0} = \int_{\sigma_2}^{\sigma_1} (dx')G(x, x')\delta\eta(x'). \tag{12}$$

According to (9), and the anticommutativity of $\delta\eta(x')$ with $\psi(x)$, we have

$$G(x, x') = i\langle(\psi(x)\bar{\psi}(x'))_+\rangle\epsilon(x, x'), \tag{13}$$

where $\epsilon(x, x') = (x_0 - x_0')/|x_0 - x_0'|$. On combining the differential equation for $\langle \psi(x)\rangle$ with (11), we obtain the functional differential equation

$$[\gamma_\mu(-i\partial_\mu - e\langle A_\mu(x)\rangle + ie\delta/\delta J_\mu(x)) + m]G(x, x') = \delta(x - x'). \tag{14}$$

An accompanying equation for $\langle A_\mu(x)\rangle$ is obtained by noting that

$$\langle j_\mu(x)\rangle = ie \operatorname{tr} \gamma_\mu G(x, x')_{x' \to x}, \tag{15}$$

in which the trace refers to the spinor indices, and an average is to be taken of the forms obtained with $x_0' \to x_0 \pm 0$. Thus, with the special choice of gauge, $\partial_\nu\langle A_\nu(x)\rangle = 0$, we have

$$-\partial_\nu^2\langle A_\mu(x)\rangle = J_\mu(x) + ie \operatorname{tr} \gamma_\mu G(x, x). \tag{16}$$

The simultaneous equations (14) and (16) provide a rigorous description of $G(x, x')$ and $\langle A_\mu(x)\rangle$.

A Maxwell field Green's function is defined by

$$\mathcal{G}_{\mu\nu}(x, x') = (\delta/\delta J_\nu(x'))\langle A_\mu(x)\rangle = (\delta/\delta J_\mu(x))\langle A_\nu(x')\rangle = i[\langle (A_\mu(x)A_\nu(x'))_+\rangle - \langle A_\mu(x)\rangle\langle A_\nu(x')\rangle]. \quad (17)$$

The differential equations obtained from (16) and the gauge condition are

$$-\partial_\lambda{}^2\mathcal{G}_{\mu\nu}(x, x') = \delta_{\mu\nu}\delta(x - x') + ie \operatorname{tr} \gamma_\mu(\delta/\delta J_\nu(x'))G(x, x),$$
$$\partial_\mu\mathcal{G}_{\mu\nu}(x, x') = 0 \ (= \partial_\nu\chi). \quad (18)$$

More complicated Green's functions can be discussed in an analogous manner. The Dirac field Green's function defined by

$$\delta_\eta{}^2\langle(\psi(x_1)\psi(x_2))_+\rangle\epsilon(x_1, x_2)_{\eta = 0} = \int_{\sigma_2}^{\sigma_1} (dx_1')\int_{\sigma_2}^{\sigma_1} (dx_2')G(x_1, x_2; x_1', x_2')\delta\eta(x_1')\delta\eta(x_2'), \quad (19)$$

may be called a "two-particle" Green's function, as distinguished from the "one-particle" $G(x, x')$. It is given explicitly by

$$G(x_1, x_2; x_1', x_2') = \langle(\psi(x_1)\psi(x_2)\bar{\psi}(x_1')\bar{\psi}(x_2'))_+\rangle\ \epsilon,$$
$$\epsilon = \epsilon(x_1, x_2)\epsilon(x_1', x_2')\epsilon(x_1, x_1')\epsilon(x_1, x_2')\epsilon(x_2, x_1')\epsilon(x_2, x_2'). \quad (20)$$

This function is antisymmetrical with respect to the interchange of x_1 and x_2, and of x_1' and x_2' (including the suppressed spinor indices). It obeys the differential equation

$$\mathfrak{F}_1G(x_1, x_2; x_1', x_2') = \delta(x_1 - x_1')G(x_2, x_2') - \delta(x_1 - x_2')G(x_2, x_1'), \quad (21)$$

where $\mathfrak{F}$ is the functional differential operator of (14). More symmetrically written, this equation reads

$$\mathfrak{F}_1\mathfrak{F}_2G(x_1, x_2; x_1', x_2') = \delta(x_1 - x_1')\delta(x_2 - x_2') - \delta(x_1 - x_2')\delta(x_2 - x_1'), \quad (22)$$

in which the two differential operators are commutative.

The replacement of the Dirac field by a Kemmer field involves alterations beyond those implied by the change in statistics. Not all components of the Kemmer field are dynamically independent. Thus, if 0 refers to some arbitrary time-like direction, we have

$$m(1 - \beta_0{}^2)\psi = (1 - \beta_0{}^2)\eta - \beta_k(-i\partial_k - eA_k)\beta_0{}^2\psi, \qquad k = 1, 2, 3, \quad (23)$$

which is an equation of constraint expressing $(1 - \beta_0{}^2)\psi$ in terms of the independent field components $\beta_0{}^2\psi$, and of the external source. Accordingly, in computing $\delta_\eta\langle\psi(x)\rangle$ we must take into account the change induced in $(1 - \beta_0{}^2)\psi(x)$, whence

$$G(x, x') = i\langle(\psi(x)\bar{\psi}(x'))_+\rangle + (1/m)(1 - \beta_0{}^2)\delta(x - x'). \quad (24)$$

The temporal ordering is with respect to the arbitrary time-like direction.

The Green's function is independent of this direction, however, and satisfies equations which are of the same form as (14) and (16), save for a sign change in the last term of the latter equation which arises from the different statistics associated with the integral spin field.

[1] Schwinger, J., *Phys. Rev.*, June 15, 1951 issue.
[2] We employ units in which $\hbar = c = 1$.

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In all of the work of the preceding note there has been no explicit reference to the particular states on σ_1 and σ_2 that enter in the definitions of the Green's functions. This information must be contained in boundary conditions that supplement the differential equations. We shall determine these boundary conditions for the Green's functions associated with vacuum states on both σ_1 and σ_2. The vacuum, as the lowest energy state of the system, can be defined only if, in the neighborhood of σ_1 and σ_2, the actual external electromagnetic field is constant in some time-like direction (which need not be the same for σ_1 and σ_2). In the Dirac one-particle Green's function, for example,

$$\begin{aligned} G(x, x') &= i\langle\psi(x)\bar{\psi}(x')\rangle, x_0 > x_0', \\ &= -i\langle\bar{\psi}(x')\,\psi(x)\rangle, x_0 < x_0', \end{aligned} \tag{25}$$

the temporal variation of $\psi(x)$ in the vicinity of σ_1 can then be represented by

$$\psi(x) = \exp\,[iP_0(x_0 - X_0)]\psi(X)\exp\,[-iP_0(x_0 - X_0)], \tag{26}$$

where P_0 is the energy operator and X is some fixed point. Therefore,

$$x \sim \sigma_1\colon G(x, x') = i\langle\psi(X)\exp\,[-i(P_0 - P_0^{\text{vac}})(x_0 - X_0)]\bar{\psi}(x')\rangle, \tag{27}$$

in which P_0^{vac} is the vacuum energy eigenvalue. Now $P_0 - P_0^{\text{vac}}$ has no negative eigenvalues, and accordingly $G(x, x')$, as a function of x_0 in the vicinity of σ_1, contains only positive frequencies, which are energy values for states of unit positive charge. The statement is true of every time-like direction, if the external field vanishes in this neighborhood.

A representation similar to (26) for the vicinity of σ_2 yields

$$x \sim \sigma_2\colon G(x, x') = -i\langle\bar{\psi}(x')\exp\,[i(P_0 - P_0^{\text{vac}})(x_0 - X_0)]\psi(X)\rangle, \tag{28}$$