$$\frac{\log N(\Delta_{K,F})}{mn} > ec_5bn \qquad (18)$$

$$> l_{r-1}(ebn^2)$$
$$> l_r(mn)$$

Coupling (18) with (9), completes the proof of Theorem 1.

* Under the auspices of the Office of Naval Research.

[1] Minkowski, H., Diophantische Approximationen, Leipzig, 1907.

[2] Rogers, C. A., "The Product of n Real Homogeneous Linear Forms," *Acta Mathematica*, **37**, 159–163 (1951).

[3] Herbrandt, M. J., "Le developpement moderne de la théorie des corps algebriques," *Mem. Sci. Math.*, **75**, 15–22 (1936).

[4] Ankeny, N. C., and Chowla, S., The Class Number of a Cyclotomic Field," PROC. NATL. ACAD. SCI., **9**, 529–532 (1949).

[5] Weyl, H., Algebraic Theory of Numbers, 1940, Princeton Univ. Press.

[6] Landau, E., *Math. Zeitschrift*, **4**, 152–162 (1919).

[7] Chowla, S., "A New Proof of a Theorem of Siegel," *Ann. Math.*, **51**, 120–122 (1950).

[8] Landau, E., Einführung in die elementare und analytische theorie der algebraischen zahlen und der ideale, 1927, Teubner, 124–135.

[9] Chevalley, C., "La Théorie de corps des classes dans les corps fini et le corps locaux," *Jr. Fac. Sci.*, Tokyo, **3**, 365–400 (1933).

[10] Hasse, H., "Führer, discriminante und verzweigungsgruppen relativ-abelscher zahlkörper," *Jr. für. Math.*, **155**, 199–220 (1926).

TYPOGRAPHICAL CORRECTION

In my paper "On the Basis Theorem for Finite Abelian Groups (Third Note)," these PROCEEDINGS, **37**, 611–614 (September, 1951), the symbols ϵ and ϵ surmounted by a bar, $\bar{\epsilon}$, were used to denote, respectively, membership and non-membership of an element in a group. Owing to a printer's error, the bar is omitted over some of the ϵ's, thus making it difficult to follow the logic of the proof. Bars should appear over the following ϵ's:

P. 611, line after "THEOREM," second ϵ.

P. 612, line 5 from bottom.

P. 613, line 3 from top.

Also, on p. 611, the bar over the ϵ on the line following the lemma should be made more clear.

JESSE DOUGLAS

ON THE BASIS THEOREM FOR FINITE ABELIAN GROUPS (THIRD NOTE)*

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1. Let G denote any finite abelian group; we wish to prove the existence of a basis $(\theta_1, \theta_2, \ldots, \theta_m)$ for G. This is to require (in additive notation) that: (*i*) every element θ of G shall be representable in the form

$$\theta = a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m \tag{1}$$

while (*ii*) (independence condition)

$$c_1\theta_1 + c_2\theta_2 + \ldots + c_m\theta_m = 0 \tag{2}$$

shall imply

$$c_i\theta_i = 0 \qquad \text{for } i = 1, 2, \ldots, m.^1 \tag{3}$$

0 denotes the identical element; small English letters, here and in the sequel, have integer values.

Because of the standard representation of any finite abelian group as direct product of subgroups of prime-power order, the case of prime-power order leads immediately to the general case; one has only to combine the bases of the component subgroups. This justifies us in supposing from now on that the order of G is p^α, p a prime, α a positive integer.

By a B-group we shall understand a subgroup of G that has a basis. Certainly B-groups exist, e.g., the cyclic subgroups of G.

Our goal is to prove the following

THEOREM. *G itself is a B-group.*

We shall use ϵ to denote membership of an element in a group, $\bar{\epsilon}$ non-membership. By $H < K$, or $K > H$, we shall mean that H is a *proper* subgroup of K: H contained in K, but $H \neq K$.

2. LEMMA. *If H is a B-group and $H < G$, then a B-group K exists such that $H < K$.*

Proof: By hypothesis, an element φ of G exists such that $\varphi \bar{\epsilon} H$. But $p^\lambda\varphi \epsilon H$ if λ is large enough, e.g., if p^λ = period of φ, so that $p^\lambda\varphi = 0$. Let $\lambda > 0$ be fixed so that p^λ is the *lowest* power of p such that $p^\lambda\varphi \epsilon H$. Then

$$p^\lambda\varphi = a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m, \tag{4}$$

where $(\theta_1, \theta_2, \ldots, \theta_m)$ is a basis of H. We suppose this basis to be without zero elements,[2] and written in a descending order (wide sense) of the respective periods:

$$p^{k_1} \geqq p^{k_2} \geqq \ldots \geqq p^{k_m}.$$

Since this is equivalent to

$$k_1 \geqq k_2 \geqq \ldots \geqq k_m,$$

each period is a multiple of all the following ones. A consequence is that $d\theta_r = 0$ implies

$$d\theta_{r+1} = 0, d\theta_{r+2} = 0, \ldots, d\theta_m = 0. \quad (5)$$

In the proof of our lemma there are two cases.

3. *Case I:* Each coefficient a_i in (4) is divisible by p: $a_i = pb_i$ for $i = 1, 2, \ldots, m$.

Then (4) can be written

$$p(p^{\lambda-1}\varphi - b_1\theta_1 - b_2\theta_2 - \ldots - b_m\theta_m) = 0. \quad (6)$$

Let

$$\omega = p^{\lambda-1}\varphi - b_1\theta_1 - b_2\theta_2 - \ldots - b_m\theta_m; \quad (7)$$

then, first, $\omega \bar{\epsilon} H$; otherwise $p^{\lambda-1}\varphi \epsilon H$, contrary to our least power hypothesis concerning p^λ.

Also

$$p\omega = 0. \quad (8)$$

From this we can infer that $c\omega \epsilon H$ implies $c\omega = 0$. For if $c\omega \neq 0$, p does not divide c; hence $xc + yp = 1$ for certain integers x, y; therefore $\omega = (xc + yp)\omega = x(c\omega) + y(p\omega) = x(c\omega)$; thus $\omega \epsilon H$, contrary to our known condition.

It can now be proved that

$$(\omega, \theta_1, \theta_2, \ldots, \theta_m) \quad (9)$$

are independent elements, and therefore form a basis of the group K which they generate.

For suppose

$$c\omega + c_1\theta_1 + c_2\theta_2 + \ldots + c_m\theta_m = 0.$$

Then $c\omega \epsilon H$, hence (by the second preceding paragraph) $c\omega = 0$; consequently also $c_1\theta_1 = 0, \ldots, c_m\theta_m = 0$, because of the independence of the θ's. Thus all the elements (9) are independent.

Accordingly, K, possessed of a basis, is a B-group. But K contains a basis of H, and therefore H itself—also the element $\omega \epsilon H$. Thus $K > H$.

4. *Case II:* At least one coefficient a_i in (4) is not divisible by p; let a_r be the first such coefficient.

We have then from (4)

$$p(p^{\lambda-1}\varphi - b_1\theta_1 - \ldots - b_{r-1}\theta_{r-1}) = a_r\theta_r + \ldots + a_m\theta_m. \quad (10)$$

Let

$$\omega = p^{\lambda-1}\varphi - b_1\theta_1 - \ldots - b_{r-1}\theta_{r-1}; \tag{11}$$

then $\omega \bar{\epsilon} H$, otherwise $p^{\lambda-1}\varphi \epsilon H$, contrary to hypothesis (cf. the text just preceding (4)).

Also, by (10),

$$p\omega = a_r\theta_r + a_{r+1}\theta_{r+1} + \ldots + a_m\theta_m, \tag{12}$$

where p does not divide a_r. Therefore integers x, y exist such that

$$xa_r + yp^{k_r} = 1. \tag{13}$$

Here p^{k_r} is the period of θ_r, so that

$$0 = p^{k_r}\theta_r. \tag{14}$$

By linear combination of (12) and (14) with the multipliers x, y, we get

$$xp\omega = \theta_r + xa_{r+1}\theta_{r+1} + \ldots + xa_m\theta_m.$$

It follows that the group K generated by

$$(\theta_1, \theta_2, \ldots, \theta_{r-1}, \omega, \theta_{r+1}, \ldots, \theta_m) \tag{15}$$

includes θ_r, therefore includes all of H.

But K also contains the element $\omega \bar{\epsilon} H$; hence $K > H$.

It remains to prove that K is a B-group. This we shall do by showing that the generators (15) of K are *independent*, and so form a basis of K.

Indeed, suppose that

$$c_1\theta_1 + c_2\theta_2 + \ldots + c_{r-1}\theta_{r-1} + c\omega + c_{r+1}\theta_{r+1} + \ldots + c_m\theta_m = 0. \tag{16}$$

Then $c\omega \epsilon H$; also $p\omega \epsilon H$, by (12). Hence p divides c: $c = dp$—otherwise $xc + yp = 1$ for certain integers x, y; $\omega = (xc + yp)\omega = x(c\omega) + y(p\omega)$; therefore $\omega \epsilon H$, contrary to the statement following (11).

Consequently, with use of (12),

$$c\omega = dp\omega = da_r\theta_r + da_{r+1}\theta_{r+1} + \ldots + da_m\theta_m. \tag{17}$$

Making this substitution in (16), we find a linear relation in the θ's wherein the term in θ_r is $da_r\theta_r$; by the independence of the θ's, it follows that

$$da_r\theta_r = 0. \tag{18}$$

Since, by (14),

$$dp^{k_r}\theta_r = 0, \tag{19}$$

we obtain with use of (13): $d\theta_r = 0$; hence, by (5),

$$d\theta_{r+1} = 0, \ldots, d\theta_m = 0. \tag{20}$$

By (17), then,

$$c\omega = 0. \tag{21}$$

From the independence of the θ's, it follows that the other terms in (16) are individually $= 0$, thus completing our proof of independence of the elements (15).

5. Let now H_0 be a B-group of *maximum* order; certainly H_0 exists, since the orders of the B-groups are bounded by the finite order of G.

We say $H_0 = G$, thus proving our main theorem.

For if $H_0 < G$, we are in obvious contradiction with our lemma.

* Recent treatments of this "fundamental theorem of abelian groups" are: by the author, these PROCEEDINGS, **37**, 359–362, 525–528 (1951); by Rado, R., *J. London Math. Soc.*, **26** (part I), 74–75 (1951). These new proofs are definitely superior in simplicity and directness to those previously given in the standard treatises on algebra and group theory.

[1] This definition allows any number of zero elements to be adjoined to or removed from a basis. Thus we may always suppose a basis to contain no zero elements—except in the case of the group consisting solely of 0, whose basis we shall regard as 0.

That the elements of a basis (without zero elements) are necessarily distinct follows immediately from the independence condition.

[2] No proof of our lemma is required in the case of the group H consisting solely of 0, for this is properly contained in every other cyclic subgroup of G, i.e., in a B-group.

HARMONIC TENSORS ON MANIFOLDS WITH BOUNDARY

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1. *Introduction*.—The boundary-value problems considered in this note relate to harmonic p-tensors on a Riemannian manifold with boundary. We state certain theorems of existence and uniqueness, which are solutions for the boundary-value problems of Dirichlet and Neumann type for harmonic p-tensors on a boundary manifold. The results which we state, with the exception of Theorem 8, were conjectured by Tucker.[7] Proofs of the theorems will appear in the *Annals of Mathematics*.

We consider alternating (skew-symmetric) covariant tensors $\varphi_{i_1 \ldots i_p}$ of rank p on a Riemannian manifold M with boundary B, and also their associated differential forms,

$$\varphi = \sum_{i_1 < \ldots < i_p} \varphi_{i_1 \ldots i_p} \, dx^{i_1} \wedge \ldots \wedge dx^{i_p}.$$