ties, but in a higher dimensional P^m (m being the lowest dimension for which this is possible), then Theorem II still holds, if one interprets Π_k as the dual cocycle class of the intersection cycle of $a(k + m - n - 1)$-dimensional plane in P^m with V^n.

[1] *Die Idee der Riemannschen Flaeche,* Weyl, H., Berlin (1913).

[2] "Ueber Schnittflaechen in speziellen Faserungen und Felder reeller und komplexer Linienelemente," Kundert, E. G., *Ann. Math.*, **54,** 215–246 (1951).

[3] "The Topological Invariants of Algebraic Varieties," Hodge, W. V. D., *Proceedings* of the International Congress of Mathematics in Cambridge, 1950.

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*A NOTE ON AUTOMORPHIC VARIETIES**

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Communicated by S. Bochner, August 29, 1952

In this note we shall make a very limited application of Theorem 1 below—namely, to the proof of the fact that compact automorphic varieties of the type considered admit only a finite number of analytic homeomorphisms.

Let D be a bounded domain in the space of n complex variables $z_1, \ldots, z_n$, and let $\Gamma = \{\gamma\}$ be a *discrete* group of analytic homeomorphisms of D on itself. The elements of Γ are necessarily countable, and we shall denote them by $\gamma_0, \gamma_1, \gamma_2, \ldots$, where γ_0 is the identity element. If we write $J_i(Z)$ for the Jacobian of γ_i, that is,

$$J_i(Z) = \frac{\partial(\gamma_i Z)}{\partial(Z)}, \tag{1}$$

then the series

$$\sum_{i=0}^{\infty} |J_i(Z)|^2 \tag{2}$$

is majorized on every closed subset of D by some convergent series of constants.[1] Hence, if $H(Z)$ is any bounded holomorphic function on D, then the Poincaré series

$$\Theta(Z) = \sum_{i=0}^{\infty} H(\gamma_i Z)\cdot[J_i(Z)]^k \tag{3}$$

also represents a holomorphic function on D, provided $k \geq 2$; and it has the periodicity property

$$\Theta(\gamma_i Z) = \Theta(Z)\cdot[J_i(Z)]^{-k}, \tag{4}$$

by virtue of which it is called an *automorphic form* of weight k. All the forms of a given weight k constitute a linear space, of dimension P_k, say, which in general may be infinite. We wish to establish the preliminary result that in any case this dimension P_k tends to infinity with k.

To show this let $A = (a_1, \ldots, a_n)$ be a point of D which is not a fixed point of any γ_i other than the identity γ_0 (the set of points not fulfilling this requirement has complex dimension at most $n - 1$); and let $\alpha, \alpha_1, \ldots, \alpha_n$ be any complex numbers, ϵ some small fixed positive number. We propose to show that for all sufficiently large weights k there exist Poincaré series (3) satisfying[2]

$$\left|\Theta(A) - \alpha\right| < \epsilon; \quad \left|\frac{\partial}{\partial z_j}\Theta(A) - \alpha_j\right| < \epsilon \qquad (j = 1, \ldots, n). \quad (5)$$

Denote by S the polycylinder $\left|z_j - a_j\right| \leq \rho, j = 1, \ldots, n$, ρ being taken small enough that S is entirely contained in D. By the remarks above we may find an integer q such that the quantities

$$c_i = \sup_S \left| J_i(z) \right| \qquad (6)$$

are less than $^1/_2$ for $i \geq q$, the series $\sum c_i^2$ being, of course, convergent. For $H(Z)$ choose any fixed function, a polynomial, say, such that $H(A) = \alpha$, $H(\gamma_i A) = 0$ for $i = 1, \ldots, q$, and such that $\frac{\partial}{\partial z_j}H(A) = \alpha_j$, $\frac{\partial}{\partial z_j} H(Z) = 0$ at $Z = \gamma_i A$ for $j = 1, \ldots, n$ and $i = 1, \ldots, q$. Such a choice is always possible, since $\gamma_i A \neq A$ for $i = 1, 2, \ldots$. Then

$$\Theta(A) = \alpha + \sum_{i=q+1}^{\infty} H(\gamma_i A)\cdot[J_i(A)]^k = \alpha + F(A),$$

where

$$F(Z) = \sum_{i=q+1}^{\infty} H(\gamma_i Z)\cdot[J_i(Z)]^k.$$

For Z in S we have

$$\left| F(Z) \right| \leq C\cdot \sum_{q+1}^{\infty} c_i^k = 2^{-k}C\cdot \sum_{q+1}^{\infty} (2c_i)^k,$$

C denoting a bound for $\left| H(Z) \right|$ on D. But for $i \geq q$ we have $2c_i < 1$, so that $(2c_i)^k \leq (2c_i)^2$, if $k \geq 2$. It follows that, for Z in S, the inequality above may be replaced by

$$\left| F(Z) \right| < 2^{-k}C\cdot \sum_{i=0}^{\infty} (2c_i)^2 = 2^{-k}C' \qquad (Z \text{ in } S), \qquad (7)$$

the constant C' depending only on $H(Z)$ and S. Obviously, we shall have $|\Theta(A) - \alpha| <$ for large k.

A simple computation shows that

$$\frac{\partial}{\partial z_j}\Theta(A) = \alpha_j + \frac{\partial}{\partial z_j}F(A) \qquad (j = 1, \ldots, n),$$

and equation 7 with the iterated Cauchy formula applied to S gives us

$$\left|\frac{\partial}{\partial z_j}F(A)\right| < \frac{C'}{2^k\rho} \qquad (j = 1, \ldots, n). \tag{8}$$

Thus our function $\Theta(Z)$, for large k, satisfies (5).

Then it is clear that, for large k, we can find n different Poincaré series $\Theta_1, \ldots, \Theta_n$, all of the same weight k, corresponding to n given systems of numbers $(\alpha_{\nu 1}, \ldots, \alpha_{\nu n})$, $\nu = 1, \ldots, n$, such that

$$\left|\frac{\partial}{\partial z_j}\Theta_\nu(A) - \alpha_{\nu j}\right| < \epsilon \qquad (\nu, j = 1, \ldots, n).$$

If ϵ is suitably small and if det $(\alpha_{\nu j}) \neq 0$, then these n forms of weight k must be linearly independent. Hence, $P_k \geq n$ for large k. But the theorem above expressed by (5) manifestly has a straightforward generalization for partial derivatives of any bounded order,[3] with which a similar argument gives immediately

THEOREM 1. *The k-genera P_k tend to infinitely with k.*

The space $M = D(\text{mod } \Gamma)$ can always be regarded as a complex analytic manifold, and we shall suppose in what follows that M is compact. We shall use the theorem just established to prove

THEOREM 2. *The total group G of analytic homeomorphisms of the automorphic variety M on itself is always finite.*[4]

The link between Theorem 1 and the present problem is afforded by the fact that G must be a complex Lie group—possibly zero dimensional, in which case it is totally discrete.[5] But G is easily seen to be compact, since D is bounded[6] and therefore if it is zero dimensional, it must be finite.

Let $\phi_1, \ldots, \phi_{P_k}$ be a basis for the holomorphic densities of weight k on M. Since the latter is compact the P_k are all finite and coincide with the numbers introduced above. For clearly every such density ϕ of weight k is the projection of an automorphic form $\tilde{\phi}$ on D of weight k (not necessarily a Poincaré series), and conversely. Now any transformation g of G transforms the ϕ_μ into new densities ϕ_μ^g, and we shall have relations of the form

$$\phi_\mu^g = \lambda_{\mu 1}(g)\cdot\phi_1 + \ldots + \lambda_{\mu P_k}(g)\cdot\phi_{P_k} \qquad (\mu = 1, \ldots, P_k), \tag{9}$$

the $\lambda_{\mu\nu}$ depending only on g, of which they would be holomorphic functions when considered as functions defined on G if the latter were positive dimensional. But since G is compact, the $\lambda_{\mu\nu}(g)$ would then be constants. We show that this is impossible for large k.

Let $t_1, \ldots, t_a$ be complex coordinates in a neighborhood of the identity of G, with all $t_j = 0$ at the identity; a is of course the dimension of G, assumed positive. Differentiating equation 9 with respect to t_j we have

$$\frac{\partial}{\partial t_j} \phi_\mu^g = 0;$$

and at $t_1 = \ldots = t_a = 0$ this is

$$\sum_{\sigma=1}^{n} \frac{\partial \phi_\mu}{\partial z_\sigma} \cdot \eta_{(j)}^{\sigma} = 0, \qquad (10)$$

where the $\eta_{(j)}^{\sigma} = \left.\frac{\partial z_\sigma}{\partial t_j}\right|_{T=0}$ are, of course, the infinitesimal generators of G, $z_1, \ldots, z_n$ being local coordinates on M. Since the $\eta_{(j)}^{\sigma}$ cannot vanish everywhere on M, we can by Theorem 1 choose k sufficiently high that equation 10 cannot hold.

* Written while the author was under contract to the Office of Naval Research.

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[2] Giraud, G., *Leçons sur les fonctions automorphes*, Gauthier-Villars, Paris, 1926, pp. 19–25.

[3] Giraud, G., *loc cit.*

[4] Andreotti, Aldo, Sopra il problema dell'uniformizzazione per alcune classi di superficie algebriche, *Rend. accad. Naz. XL*, Series IV, **II**, 16 (1951).

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[5] Bochner, S., and Montgomery, D., "Groups on Analytic Manifolds," *Ibid.*, **48**, No. 3, 659–669 (July, 1947).

[6] Bochner, S., *loc cit.*