at California Institute of Technology, contains before cooking, for 100 parts water, 0.885 part Agar; 10.6 parts molasses; 8.85 parts yellow cornmeal; 1.24 parts dry Brewer's yeast; and 0.53 part propionic acid. This is cooked to remove about 29 parts water and poured into bottles.

[10] Bowen, V. T., *J. Exptl. Zool.*, **118**, 509–530 (1951).

CONFORMAL RELATIVITY

BY RICHARD L. INGRAHAM

INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY

Communicated by O. Zariski, August 2, 1952

If one keeps invariant angle but drops invariant (four-dimensional) length from an essentially Einsteinian description of the world, one gets Conformal Relativity. This theory, skeletonized below, will be published elsewhere.

The basic idea is to let the invariance of the light cone alone (without the additional requirement that there exist an invariant length) define the class of *preferred systems* (i.e., those observers for whom the nature laws are "in their simplest form"). The field equations of the resulting theory thus exhibit *invariance in form* under the whole conformal group in local space-time, interpreted as coordinate transformation group. Accordingly, we define the *Special Theory of Conformal Relativity* to be the study of the ordinary conformal ("conformal flat") space C_4 with angle-defining form of signature $(+++-)$ 1 when the four-dimensional manifold is interpreted as space-time. It treats the kinematical relationship between equivalent observers and the behavior of measuring apparatus in the absence of all "force" fields.

The mathematical techniques needed are classical, and some of the results are already in the literature. The main result is due to J. Haantjes[1] who in 1940 discovered that the conformal group represents transformations between all observers (cartesian systems in different R_4's) in *uniform* relative acceleration. All relative velocities are bounded above by light velocity, which is of course the same for all observers. For details on the non-invariance of length and rest mass,[2] see Haantjes *loc. cit.* Other results proper to the Special Theory are, e.g. the behavior of meter sticks and clocks under conformal transformations, the composition law of relative accelerations, etc.

An *event* of this theory is defined to be a hypersphere (center x^m ($m = 1, \ldots 4$), radius R), the subclass of nullspheres[3] ($R = 0$), identified with their center points x^m, corresponding 1-1 with the events of ordinary relativity. Hexaspherical coordinates,[4] the preferred systems of C_4, refer these events to six *real* independent hyperspheres as coordinate surfaces. Thereby points (nullspheres) of C_4 have hexaspherical coordinates Z^a ($a = 0, 1, \ldots 5$) satisfying a quadratic equation

$$S_{ab}Z^aZ^b = 0. \tag{1}$$

The subgroup of the group of all *constant linear* transformations of the hexaspherical coordinates of events $Z^{a'} = P^a{}_bZ^b(a' = a)$ which takes points of C_4 into points of C_4:

$$S_{ab}P^a{}_cP^b{}_d = hS_{cd} \quad \text{(for some } h\text{)} \tag{2}$$

is just the 15 parameter group of conformal (coordinate) transformations of points of C_4.[4] This is then the fundamental group of the Special Theory.

The physical interpretation of the hypersphere-event is the specification of position x^m in space-time and some standard reference length or *gauge* R (real or imaginary if space- or time-like, resp.) at that position. We have to do then, in Conformal Relativity, with a five-dimensional continuum, *the space-time-gauge continuum.* Gauge at a point in general changes under a conformal transformation, and the proportionate change is *not* the same for all gauges, unlike certain previously considered gauge theories.[5] Only zero gauge is preserved.

Since the hexaspherical coordinates Z^a of events in C_4 define an ordinary real five-dimensional projective space P_5, everything can be equivalently phrased in the five-dimensional projective language in a well-known way.[6] I.e. (we give it for later convenience), the geometry of an ordinary conformal space C_n (of signature Sig. g_{mn} say) is identical with that of an ordinary real $n + 1$-dimensional projective space P_{n+1} provided with a special quadric S_{ab} of signature $(-1, \text{Sig. } g_{mn}, +1)$ representing points—identified with nullspheres—of X_n (cf. Ref. 1), whose group is restricted to those collineations taking the quadric into itself (cf. Ref. 2).

The Special Theory cannot properly be called a relativity mechanics, for there are no laws of motion under various forces. These are given by the General Theory of Conformal Relativity. It is defined as the study of a general ("curvilinear") conformal space K_4 (to be defined infra) with angle-defining form of signature $(+++-)$ 1 over the X_5 of space-time-gauge *which is subjected to the one global condition of minimum total (i.e., integrated) curvature.* In addition to the kinematical results known from the Special Theory, one gets as extremal equations a set of nature laws describing various "force" fields.

We define: *A general ("curvilinear") conformal space K_n is an $n + 1$-dimensional manifold X_{n+1} with ordinary conformal spaces C_n of the same signature as local spaces.* In the $n + 1$-dimensional projective language (cf. the corresponding definition of C_n supra), it is thus a curvilinear real

$n + 1$-dimensional projective space H_{n+1}[7] (homogeneous coordinates X^μ say) with special quadric $S_{\lambda\kappa}(X^\mu)$ homogeneous of some degree.

For the purposes of the General Theory, we then start with a K_4 whose manifold X_5 is interpreted as the space-time-gauge continuum and whose quadric $S_{\lambda\kappa}$ is of signature $(-, +++-, +)1$. In addition it is by hypothesis linearly connected: we are given a projective connection $\Gamma_{\mu\lambda}{}^\kappa$. $\Gamma_{\mu\lambda}{}^\kappa$ must be homogeneous of degree -1 so that covariant differentiation wrt it takes projective quantities of H_{n+1} into projective quantities of H_{n+1}. Also in order that the total curvature (given by (3) infra) exist as a projective invariant, the degree of $S_{\lambda\kappa}$ must be chosen -2. Let the curvature tensor of Γ be $N_{abc}{}^d$, the Ricci tensor $N_{bc} \equiv N_{abc}{}^a$, the Gaussian curvature $N \equiv S^{ab}N_{ab}$. Then the condition of minimum total curvature requires

$$\delta \int N\sqrt{S}(dX) = 0 \qquad (S \equiv \text{Det } S_{\lambda\kappa},\ (dX) \equiv dX^\circ\, dX^1 \ldots dX^5) \qquad (3)$$

As in Einsteinian relativity, this global condition is all that is required of the geometry.

The connection Γ and the quadric S are independently variable in (3). It turns out that Γ is necessarily not only symmetric but must be the Christoffel symbols of S.[8] *These extremal equations describe Einstein-type gravitation, Maxwell-type electromagnetism, and several mesons* (*tentatively interpreted*). Thus a *unified* field theory of force fields necessarily results from dropping invariant length from an essentially Einsteinian description of nature. The mesons are vector and scalar wrt a group transforming four-position variables; they are particles of integral spin and non-zero rest mass (defined *a posteriori* of course[9]) in the language of particle theory. The electromagnetic and mesic theories thereby given are approximately linear in regions of tenuous energy, becoming essentially non-linear in regions where energy is dense. The interpretation (classical, quantum ...) is left open. The field equations are covariant against the group of all holonomic and non-holonomic transformations in H_5, that is, under arbitrary projective coordinate transformations in the local P_5's. In the four-dimensional language, they express the vanishing of a hexaspherical tensor which is formed out of the curvature in the familiar way. The field equations are also all conformally *invariant in form* (—i.e., maintain their "simplest form") under the whole conformal group or, kinematically, with respect to the class of all observers in uniform relative acceleration.

Conformal Relativity's relation to the former relativities, projective and affine (or "general"), with which it forms a generic class, will be treated in detail in the article to appear; here only a few of the more important remarks can be made.

1. In many ways, projective relativity[10] (or the Kaluza theory[11]) reveals itself as a provisional stage on the way to Conformal Relativity in

that features which were anomalous or arbitrary in the former are only explained or uniquely fixed when we come to the latter; e.g.:

(*a*) The point of view advanced in the present theory is that the appearance of projective geometry in unified field theory arises naturally *only* in handling the conformal geometry of four-dimensions in the framework of a five-dimensional projective tensor calculus.

(*b*) The anomalous 5×5 quadric of the projective relativity—Kaluza theory, when completed to the 6×6 quadric (1) of this theory, has the well-defined geometric role of defining the point-events among the class of hypersphere-events. Thus incidentally, any skew part of S_{ab} would be geometrically meaningless here, and is therefore excluded from consideration.[12]

(*c*) There is no need for any arbitrary restriction on spatial symmetry of the cylinder condition type to eliminate an unwanted dimension— space-time-gauge fills out a whole X_5.

(*d*) *A priori* criteria were lacking to define completely the quadric's signature in projective relativity.[13] But it is a fact that the quadric's signature in this theory is fixed as $(-1, \text{Sig. } g_{mn}, +1)$ by the signature of space-time plus the demand that the local hexaspherical coordinates be real—i.e., that the local spaces be real P_5's.

2. Fundamental is the fact that only in this theory do we get the identity of the class of all observers "indistinguishable with respect to the light cone" (kinematically, all those in uniform relative acceleration) and the class of all observers "indistinguishable with respect to the nature laws" (preferred systems). Calling these classes C_1 and C_2, respectively, in Conformal Relativity $C_1 = C_2$. In projective relativity, C_1 is not a subset of C_2, C_2 is not a subset of C_1. In affine ("general") relativity, $C_2 \subset C_1$, but $C_2 \neq C_1$. In both latter relativities, the intersection $C_1 \cap C_2$ is a class of observers connected by orthogonal transformations, so that an invariant length figures in both theories.

[1] Haantjes, J., *Proc. Ned. Akad. v. Wet.*, **43,** 3 (1940).

[2] Haantjes treats only the classical Maxwell theory, where rest mass is of course represented by a number, *a priori* given, attached to each particle.

[3] In view of the signature $(+++-)$ 1, nullspheres are light cones, viewed as real loci, and centers x^m are vertices.

[4] For details on polyspherical coordinates, see Klein, F., *Vorlesungen über Höhere Geometrie*, 3rd ed., Springer, Berlin, 1926.

[5] Weyl, H., *Raum. Zeit. Materie*, 4th ed., Springer, Berlin, 1921.

[6] Klein, F., *Math. Ann.*, **5,** 257 (1872).

[7] For details on the curvilinear projective geometries H_n, see Schouten, J., and Haantjes, J., *Comp. Math.*, **3,** 1 (1936).

[8] Cf. Schrödinger, E., *Proc. Roy. Irish Acad.*, **51A,** 63 (1947) for the four-dimensional affine case.

[9] Cf. Einstein, Infeld, Hoffmann, *Ann. Math.*, **39,** 65 (1938). And it is worth remarking that the non-invariance of this rest mass under conformal transformations will be

the explanation offered by Conformal Relativity of the apparent multiplicity of certain kinds of integral spin particles encountered experimentally.

[10] Veblen, O., and Hoffman, B., *Phys. Rev.*, **36,** 810 (1930).

[11] Kaluza, T., *Sitzber. Preuss. Akad. Wiss.*, 966 (1921).

[12] In contrast to Einstein, A., *Ibid.*, 414 (1925), or (8).

[13] Pauli, W., *Ann. Physik*, **V18,** 305 (1933).