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[1] Winge, O., and Roberts, C., *Compt. rend. trav. lab. Carlsberg, Ser. physiol.*, **24**, 263–315 (1948).

[2] Spiegelman, S., Sussman, R. R., and Pinska, E., PROC. NATL. ACAD. SCI., **36**, 591–606 (1950).

[3] Spiegelman, S., DeLorenzo, W. F., and Campbell, A. M., *Ibid.*, **37**, 513–524 (1951).

[4] Spiegelman, S., Lindegren, C. C., and Lindegren, G., *Ibid.*, **31**, 95–102 (1945).

[5] Winge, O., and Roberts, C., *Compt. rend. trav. lab. Carlsberg, Ser. physiol.*, **24**, 341–346 (1949).

[6] Adams, A. M., *Can. J. Res., C*, **27**, 179–189 (1949).

[7] Winge, O., and Roberts, C., *Compt. rend. trav. lab. Carlsberg, Ser. physiol.*, **25**, 35–83 (1950).

[8] Ephrussi, B., *Coll. Internal., Centre de la Recherche Scientifique*, **8**, 165–180 (1949).

[9] Winge, O., *Wallerstein Lab. Communic.*, **15**, 21–42 (1952).

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# AN ALGEBRA OF UNBOUNDED OPERATORS

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This paper is a direct extension of the investigation by M. H. Stone[1] of commutative self-adjoint operator algebras by means of function space techniques. (See also later work by Yosida.[2]) If $\mathfrak{A}$ is such an operator algebra, which is strongly closed, $\mathfrak{A}$ is isomorphic to all continuous complex functions on a certain Hausdorff space $X$, and to each $f$ of a certain class of unbounded complex functions on $X$ we assign an unbounded normal operator $\bar{T}_f$. The properties of $\bar{T}$ are then studied in detail. Although the proof is omitted for lack of space, each normal operator $A$ is of the form $\bar{T}_f$, where $\mathfrak{A}$ is the algebra generated by $(1 + AA^*)^{-1}A$ and $(1 + AA^*)^{-1}$.

Besides the study of $\bar{T}$, a generalization of a theorem of M. H. Stone[5] on extremally disconnected spaces is proved, and the spectral theorem for unbounded normal operators is a simple consequence of the investigation. The fundamental tool, Lemma 1.1 on monotone convergence, is essentially due to Bochner and Ky Fan.[4]

*1. Functional Representation.*—Throughout $H$ will be a fixed Hilbert space and $\mathfrak{A}$ will be a commutative algebra of bounded operators on $H$

which contains the identity 1, is self-adjoint (if $A \epsilon \mathfrak{A}$ then the adjoint $A^* \epsilon \mathfrak{A}$) and is strongly closed (i.e., if $\{A_n\}_n$ is a directed set in $\mathfrak{A}$ and $A$ is a bounded operator such that $||A_nx - Ax|| \rightarrow _n 0$ for each $x \epsilon H$ then $A \epsilon \mathfrak{A}$). Convergence will always be Moore-Smith convergence unless specifically noted otherwise. We write $A \geqq 0$ to mean $A$ is a non-negative operator.

*1.1.* THEOREM. *If $\{A_n\}_n$ is a monotone increasing directed set of Hermitian operators (i.e., if $m > n$ then $A_m \geqq A_n$) and if for some real $k$, $k \geqq ||A_n||$ for all $n$, then $\{A_n\}_n$ converges strongly to its least upper bound $B$. Consequently any bounded set of Hermitian members of $\mathfrak{A}$ has a least upper bound which belongs to $\mathfrak{A}$.*

*Proof:* If $A$ is any bounded non-negative operator then $(Ax, y)$ is an inner product so that, by the Schwartz inequality, $|(Ax, y)|^2 \leqq (Ax, x)(Ay, y)$. Consequently, $\sup\{|(Ax, y)|^2: ||y|| \leqq 1\} \leqq (Ax, x)||A||$, and since $||Ax||^2 = |(Ax, Ax/||Ax||)|^2 \leqq \sup\{|(Ax, y)|^2: ||y|| \leqq 1\}$, it is true that $||Ax||^2 \leqq (Ax, x)||A||$. Considering now the increasing $\{A_n\}_n$, it is clear from the polarization identity that $(A_nx, y)$ converges, for each $x$, $y$, and since the limit is a bounded bilinear function of $x$ and $y$, there is a bounded operator $B$ such that $(A_nx, y) \rightarrow _n (Bx, y)$ for all $x$ and $y$. Since $B - A_n \geqq 0$, from above, $||(B - A_n)x||^2 \leqq ((B - A_n)x, x)||B - A_n||$ and this converges to zero. The rest of the theorem is simple in view of the functional representation of $\mathfrak{A}$.

If $X$ is the set of multiplicative linear functionals on $\mathfrak{A}$ with the weak * topology then the map $F$, defined by $F(A)(p) = p(A)$ for $p \epsilon X$, $A \epsilon \mathfrak{A}$, carries $\mathfrak{A}$ isomorphically[6] onto the algebra $\mathfrak{C}$ of complex valued continuous functions on $X$ in such a way that $F(A^*)$ is the conjugate function $F(A)^{\sim}$. Non-negative operators correspond to non-negative functions. The inverse map to $F$ will always be denoted $T$, so that for $f \epsilon \mathfrak{C}$, $T_f$ is that operator such that $F(T_f) = f$. Observe that $E$ is a projection if and only if $F(E)$ is the characteristic function of an open and closed set.

If $U$ is any open set in $X$ then the set of all non-negative continuous functions $f$ which are zero outside $U$ and are everywhere $\leqq 1$, has, in view of the preceding theorem, a least upper bound $g$ in $\mathfrak{C}$. It is easy to see that $g = 0$ on $X \backslash U$, and 1 on $U$ and hence 1 on the closure $\bar{U}$ of $U$. Since $g$ is continuous, $\bar{U}$ is both open and closed. A space such that the closure of every open set is open is *extremally disconnected.*

*1.2 X is extremally disconnected, and finite linear combinations of the characteristic functions of open and closed sets are dense in $\mathfrak{C}$.*

*2. Extremally Disconnected Spaces.*—If $S$ is a Borel subset of a topological space there is an open set $U$ such that the symmetric difference $(S\backslash \bar{U}) \cup (U\backslash S)$ is of the first category. If the space is extremally disconnected the set $U$ may be taken open and closed. If the space is compact, so that each open set is of second category, then the corresponding open-closed set $U$ is unique.

*For each Borel subset* $S$ *of* $X$, $S'$ *will denote the unique* open and closed set such that $[(S\setminus S') \cup (S'\setminus S)] \in$ Cat I.

Given a Borel set $S$, on considering the open sets $S'$ and $X\setminus S'$, it is clear that $p \in S'$ if and only if for some neighborhood $U$ of $p$ the set $U\setminus S$ is of Cat I. Using this criterion it is easy to see:

*2.1. If* $R$ *and* $S$ *are Borel sets,* $(R \cup S)' = R' \cup S'$, $(R \cap S)' = R' \cap S'$, $R'' = R'$ *and* $(X\setminus R)' = X\setminus R'$.

M. H. Stone has proved the following theorem for the case $Y$ equals a closed interval of real numbers. A Borel function is a function such that the inverse of a Borel set is a Borel set.

*2.2* THEOREM. *If* $f$ *is a Borel function on* $X$ *to the compact metric space* $Y$ *then there is a continuous function* $g$ *on* $X$ *to* $Y$ *such that* $f = g$ *save on a set of the first category.*

*Proof:* A subdivision $\mathfrak{F}$ of $Y$ is a finite disjoint family of Borel sets whose union is $Y$, and *mesh* $\mathfrak{F}$ is the maximum diameter of a member of $\mathfrak{F}$. Choose a sequence $\{\mathfrak{F}_n\}_n$ of subdivisions, each a refinement of its predecessor (each member of $\mathfrak{F}_{n+1}$ is a subset of a member of $\mathfrak{F}_n$) with *mesh* $\mathfrak{F}_n < 1/n$. For each $\mathfrak{F}_n$ choose a function $g_n$ as follows: for each $S \in \mathfrak{F}_n$, $g_n$ is constant on $f^{-1}(S)'$, and on this set takes a value belonging to $S$. Then $g_n$ is continuous and $|g_n(p) - f(p)| \leqq$ mesh $\mathfrak{F}_n < 1/n$ save on a set of first category. The sequence $g_n$ is a Cauchy sequence since $|g_n(p) - g_{n+m}(p)| \leqq$ mesh $\mathfrak{F}_n$, and converges to a continuous function $g$, and $|g(p) - g_n(p)| \leqq$ mesh $\mathfrak{F}_n$. The set of all $p$ where $|g(p) - f(p)| > 2/n$ is consequently of the first category,† and the theorem follows.

*Let* $\bar{\mathfrak{C}}$ *be the set of all continuous functions on* $X$ *to the complex sphere which are* $\infty$ *only on a non-dense set.* $\bar{\mathfrak{C}}$ is an algebra, if we agree that $fg$ and $f + g$ are those continuous functions which agree with the product and sum save on a set of Cat I. Clearly $\bar{\mathfrak{C}}$ is isomorphic (algebraically) to the algebra of all complex Borel functions modulo the ideal of functions vanishing outside a set of the first category.

*3. Extension of T.*—For $f \in \bar{\mathfrak{C}}$ let $\mathfrak{F}$ be the set of all characteristic functions of open and closed subsets which are contained in $X\setminus f^{-1}(\infty)$. The set $\mathfrak{F}$ is directed by $\geqq$, and, using 1.1, $T_e$ converges strongly, for $e \in \mathfrak{F}$, to the identity. For $e \in \mathfrak{F}$, $ef \in \mathfrak{C}$.

*Let* $\bar{T}_f x$ *be* $\lim \{T_{ef}x : e \in \mathfrak{F}\}$, *whenever this limit exists.* Observe that if $e$ and $d$ belong to $\mathfrak{F}$ and $e \geqq d$ then $\|T_{ef}x\|^2 = (T_{e|f|^2}x, x)$ and $(T_{(e-d)|f|^2}x, x) = \|T_{ef}x\|^2 - \|T_{df}x\|^2$, so that, applying the Cauchy criterion for convergence, either $\|T_{ef}x\|$ converges for $e \in \mathfrak{F}$ to $\infty$, or it converges to a finite number, in which case $T_{ef}x$ converges to $\bar{T}_f x$.

If $d \in \mathfrak{F}$ then $T_d x$ belongs to the domain of $\bar{T}_f$, for $\|T_{ef}T_d x\|$ is bounded by $\|T_{fd}x\|$. It follows that the domain of $\bar{T}_f$ is dense in $H$. Moreover,

$\bar{T}_f T_d x = \lim\{T_{ef} T_d x : e \in \mathfrak{F}\} = \lim \{T_e T_{fd} x : e \in \mathfrak{F}\} = T_{fd} x$, for all $x \in H$. It is also straightforward to see that $T_d \bar{T}_f x = T_{df} x$ for all $x$ in the domain of $\bar{T}_f$.

*3.1.* MONOTONE CONVERGENCE THEOREM. *If $\{g_n\}_n$ is a directed set of members of $\mathfrak{C}$ which approach $f \in \bar{\mathfrak{C}}$ monotonically, in the sense that $|f - g_n|$ is monotone decreasing with 0 as lower bound in $\bar{\mathfrak{C}}$, then $T_{g_n} x$ converges to $\bar{T}_f x$ for each $x \in$ domain $\bar{T}_f$, and $\|T_{g_n} x\|$ converges to $\infty$ for $x$ not an element of domain $\bar{T}_f$.*

*Proof:* First, for each $e \in \mathfrak{F}$, $\|(T_{ef} - T_{eg_n})x\|^2 = (T_{e(f-g_n)(f-g_n)^\sim} x, x)$ converges to zero with $n$, because $e|f - g_n|^2$ is a monotonic decreasing directed set and the corresponding operators converge strongly to zero by 1.1. Next, suppose $x \in$ *domain* $\bar{T}_f$, and that $g$ and $h$ are members of $\mathfrak{C}$ such that $|f - g|^2 \geqq |f - h|^2$. Then $\|\bar{T}_f x - T_g x\|^2 = \lim \{\|T_{ef} x - T_{eg} x\|^2 : e \in \mathfrak{F}\} \geqq \lim \{(T_{e(f-h)(f-h)^\sim} x, x) : e \in \mathfrak{F}\} = \|\bar{T}_f x - T_h x\|^2$. It follows that if $m \geqq n$ and $d \geqq e$ then $\|\bar{T}_f x - T_{dg_m} x\| \geqq \|\bar{T}_f x - T_{eg_n} x\|$. Now, for $x \in$ *domain* $\bar{T}_f$, choose $e \in \mathfrak{F}$ so $T_{ef} x$ is near $\bar{T}_f x$, then $n$ so that if $m \geqq n$ then $T_{eg_m} x$ is near $T_{ef} x$ and hence near $\bar{T}_f x$. As $e$ converges over $\mathfrak{F}$ to 1, the distance $\|\bar{T}_f x - T_{eg_m} x\|$ decreases, so that $\|\bar{T}_f x - T_{g_m} x\|$ is small for $m \geqq n$. If $x$ is not an element of *domain* $\bar{T}_f$, choose $e \in \mathfrak{F}$ so $T_{ef} x$ is large whence for some $n$, if $m \geqq n$, $\|T_{eg_m} x\|$ is large, and since this norm is monotonic increasing in $e$, $\|T_{g_m} x\|$ is large.

A particular consequence of the theorem is that, in computing $\bar{T}_f x$, $\mathfrak{F}$ may be replaced by any subset $\mathcal{E}$ of $\mathfrak{F}$ which is directed by $\geqq$ and such that $T_e$, for $e \in \mathcal{E}$, converges strongly to the identity.

*3.2.* THEOREM. *$\bar{T}_f$ is normal, and its adjoint operator is $\bar{T}_{f^\sim}$.*

*Proof:* Since $x \in$ *domain* $\bar{T}_f$ if and only if $\|T_{ef} x\|$ is uniformly bounded for $e \in \mathfrak{F}$, and since $\|T_{ef^\sim} x\| = \|T_{ef} x\|$, it is clear that *domain* $\bar{T}_{f^\sim}$ = *domain* $\bar{T}_f$. Also $\|\bar{T}_f x\| = \lim \{\|T_{ef} x\| : e \in \mathfrak{F}\} = \lim \{\|T_{ef^\sim} x\| : e \in \mathfrak{F}\} = \|\bar{T}_{f^\sim} x\|$, so it remains to prove that $\bar{T}_{f^\sim}$ is the adjoint of $\bar{T}_f$. Suppose $(\bar{T}_f x, y) = (x, y*)$ for each $x \in$ *domain* $\bar{T}_f$. Then $(\bar{T}_f x, y) = \lim \{(T_{ef} x, y) : e \in \mathfrak{F}\} = \lim \{(x, T_{ef^\sim} y) : e \in \mathfrak{F}\}$, and it will follow that $y* = \bar{T}_f y$ if it can be shown that $\|T_{ef^\sim} y\|$ is bounded for $e \in \mathfrak{F}$. Let $x = T_{df^\sim} y$ where $d \in \mathfrak{F}$. Then $(T_{df^\sim} y, y*) = \lim\{(T_{ef} T_{df^\sim} y, y) : e \in \mathfrak{F}\} = \lim \{(T_e T_{dff^\sim} y, y) : e \in \mathfrak{F}\} = \|T_{df^\sim} y\|^2$. Dividing by $\|T_{df^\sim} y\|$, it is clear that $\|T_{df^\sim} y\| \leqq \|y*\|$, and the theorem is established.

*3.3.* THEOREM. *If $\{x_n\}_n$ is a sequence in domain $\bar{T}_f$, if $\|x_n - x\| \to_n 0$ and $\|\bar{T}_f x_n\|$ is bounded, then $\bar{T}_f x_n$ converges weakly to $\bar{T}_f x$. If $x$ is not an element of domain $\bar{T}_f$ then $\|\bar{T}_f x_n\| \to_n \infty$.*

*Proof:* Suppose $\|\bar{T}_f x_n\| \leqq k$ for each $n$. Since for $e \in \mathfrak{F}$, $\|T_{fe} x_n\| \leqq \|\bar{T}_f x_n\| \leqq k$ then, letting $n \to \infty$, $\|T_{fe} x\| \leqq k$ for each $e \in \mathfrak{F}$ and hence $x \in$ *domain* $\bar{T}_f$. Since $\bar{T}_f x_n$ is a bounded sequence, weak convergence to $\bar{T}_f x$ will be proved if $(\bar{T}_f x_n, y) \to_n (\bar{T}_f x, y)$ for $y$ belonging to a dense subset of $H$. Let $y = T_e z$, for $e \in \mathfrak{F}$. Then $(\bar{T}_f x_n, T_e z) = (T_e \bar{T}_f x_n, T_e z) =$

$(T_{ef}x_n, T_e z) \to_n (T_{ef}x, T_e z) = (T_e\bar{T}_f x, T_e z) = (\bar{T}_f x, T_e z)$. To prove the second statement, if $e \in \mathfrak{F}$ so that $\|T_{ef}x\|$ is large, then for $n$ large, $\|T_{ef}x_n\|$ is large, and $\|T_{ef}x_n\| \leqq \|\bar{T}_f x_n\|$.

This theorem strengthens the result that the graph of $\bar{T}_f$ is closed.

We now show the closure of the graph of $\bar{T}_f\bar{T}_g$ is the graph of $\bar{T}_{gf}$. Let $\mathcal{E}$ be the set of characteristic functions of all open and closed sets which are disjoint from $f^{-1}(\infty) \cup g^{-1}(\infty)$. Then $\bar{T}_f x$, $\bar{T}_g x$ and $\bar{T}_{fg}x$ can each be computed as $\lim \{T_{ef}x : e \in \mathcal{E}\}$, etc. If $\bar{T}_g x \in$ domain $\bar{T}_f$ then for $e \in \mathcal{E}$, $\bar{T}_f\bar{T}_g x = \lim \{\bar{T}_f T_{eg}x : e \in \mathcal{E}\} = \lim \{\bar{T}_f T_e T_{eg}x : e \in \mathcal{E}\} = \lim \{T_{efg}x : e \in \mathcal{E}\}$, and since this limit exists, it is $\bar{T}_{fg}x$. The graph of $\bar{T}_f\bar{T}_g$ is then a subset of the graph of $\bar{T}_{fg}$. For $x \in$ domain $\bar{T}_{fg}$, choose $e \in \mathcal{E}$ so that $x$ is near $T_e x$ and $T_{efg}x$ is near $\bar{T}_{fg}x$. Then $\bar{T}_g T_e x = T_e T_{eg} x$ belongs to domain $\bar{T}_f$, and since $\bar{T}_f\bar{T}_g T_e x = T_{efg}x$ the pair $\langle T_e x, \bar{T}_f\bar{T}_g T_e x\rangle$ is near the pair $\langle x, \bar{T}_{fg}x\rangle$. The same sort of argument shows that the graph of $\bar{T}_{f+g}$ is the closure of the graph of $\bar{T}_f + \bar{T}_g$ and the following theorem is established.

3.4. THEOREM. *The algebra* $\bar{\mathfrak{C}}$ *is isomorphic to an algebra* $\bar{\mathfrak{A}}$ *under* $\bar{T}$, *in the sense that the graphs of* $\bar{T}_{fg}$ *and* $\bar{T}_{f+g}$ *are, respectively, the closures of the graphs of* $\bar{T}_f\bar{T}_g$ *and* $\bar{T}_f + \bar{T}_g$.

If $f \in \bar{\mathfrak{C}}$ and the set of zeros of $f$ contains an open and closed set then $ef = 0$ for the characteristic function $e$ of this set. Each $x$ belonging to the range of $T_e$ maps under $\bar{T}_f$ into zero, so that $\bar{T}_f$ can have no inverse. If the zeros of $f$ are non-dense we will now show that $\bar{T}_{1/f}\bar{T}_f x = x$ for each $x \in$ *domain* $\bar{T}_f$. If $e$ is the characteristic function of an open and closed set disjoint from $(1/f)^{-1}(\infty)$, $T_{e(1/f)}\bar{T}_f x = T_{e(1/f)}T_e\bar{T}_f x = T_e x$. Consequently, the norm of this vector is bounded, for such an $e$, by $\|x\|$ and $\bar{T}_f x \in$ *domain* $\bar{T}_{1/f}$, and the preceding theorem completes the proof.

3.5. $\bar{T}_f$ *has an inverse if and only if the zeros of f are non-dense, and in this case* $(\bar{T}_f)^{-1} = \bar{T}_{1/f}$.

4. *The Spectral Resolution for* $\mathfrak{A}$.—The following form of the spectral theorem is suggested by the measure theoretic form used by Halmos.[5] *For each Borel set U in X let* $\mu(U) = T_e$, *where e is the characteristic function of the open and closed set* $U'$ *such that the symmetric difference* $(U' \setminus U) \cup (U \setminus U')$ *is of first category.* The projection valued set function $\mu$ is the *spectral measure* for $\mathfrak{A}$ (or for $\bar{\mathfrak{A}}$). If $\mathcal{V}$ is a family of Borel subsets of $X$ we agree that $\Sigma\{\mu(U) : U \in \mathcal{V}\}$ is to be the limit in the strong operator topology of the sums over finite subfamilies of $\mathcal{V}$, the limit being taken in the direction of increasing subfamilies.

4.1. $\mu$ *is additive over countable disjoint families of Borel sets, and over arbitrary disjoint families of open sets.*

*Proof:* If $\mathcal{V}$ is a disjoint family of Borel sets and $\mathcal{V}'$ is the family of corresponding open and closed sets then the finite sums defining $\Sigma\{\mu(U) : U \in \mathcal{V}\}$ are a monotone increasing directed set of projections bounded by the identity, and consequently converge in the strong operator topology to their least upper bound. This least upper bound is the correspondent under $T$ of the characteristic function $e$ of the closure of $\cup \{U: U \in \mathcal{V}'\}$. If the members of $\mathcal{V}$ are open, or if $\mathcal{V}$ is countable, $T_e$ is $\mu(\cup \{U: U \in \mathcal{V}'\})$.

For a Borel set $U$ in $X$ let $\mathcal{V}$ be the family of all open sets which contain $U$. Then $\mathcal{V}$ is directed by $\subset$, and we will show that $\mu(U)$ is the limit in the strong operator topology of $\mu(V)$ for $V \in \mathcal{V}$. First, in case $U$ is non-dense, $\lim \{\mu(V): V \in \mathcal{V}\}$ is 0, because of 1.1, and for each $x \in H$, $||\mu(V)x|| \to_V 0$. Next, if $U$ is the union of a sequence of non-dense sets, we may, by the familiar "$\epsilon/2^n$" argument, again see that $||\mu(V)x|| \to_V 0$ for each $x \in H$. The result, for arbitrary Borel set $U$, is now immediate. By complementation it follows that $\mu(U)$, for each Borel set $U$, is the strong limit of $\mu(C)$ for compact subsets $C$ of $U$ directed by $\supset$. A set function $\nu$, on the Borel sets of $X$ to a topological space $L$, such that $\nu(U)$ is the limit of $\nu(V)$ for $V$ open and containing $U$ is *regular* relative to the topology of $L$. The following theorem is due to Halmos.[5]

*4.2. $\mu$ is regular relative to the strong topology for $\mathcal{A}$. If $G$ is a function on $\mathcal{A}$ to a topological space which is continuous relative to the strong topology then $G\mu$ is regular.*

For $f \in \bar{\mathcal{C}}$ and $\nu$ a finitely additive function on the Borel sets of $X$ to a linear topological space we define $\int f d\nu$ as follows. (The integral depends on the topology assigned the linear topological space.) If $f$ is the characteristic function of a Borel set $U$ then $\int f d\nu = \nu(U)$. A *simple* function is a finite linear combination of characteristic functions of Borel sets, and its integral is defined accordingly. For $f \in \mathcal{C}$, $f \geqq 0$, let $\mathcal{G}$ be the set of all simple functions $g$ such that $o \leqq g \leqq f$. Then $\mathcal{G}$ is directed by $\geqq$, and $\int f d\nu$ is the limit of $\int g d\nu$ for $g \in \mathcal{G}$. The integral of an arbitrary real function belonging to $\bar{\mathcal{C}}$ is the difference of the integrals of the positive and negative parts, and the integral of a complex function is (*integral of real part*) + $i$ (*integral of imaginary part*). It is straightforward to verify that $T_f = \int f d\mu$ for $f \in \mathcal{C}$. (The limit defining the integral is taken in the norm topology for $\mathcal{A}$.)

For $x \in H$ let $\mu^x$ be the set function such that for each Borel set $U$, $\mu^x(U) = \mu(U)x$. Then $\mu^x$ has values in $H$. By using the theorem 3.1 on monotone convergence, and noticing that if $x \in$ *domain* $T_f$ then $x$ belongs to the domain of the operators corresponding to each of the four parts of $f$, one obtains:

4.3. SPECTRAL THEOREM. *For $f \in \mathcal{C}$, $T_f = \int f d\mu$. For $f \in \bar{\mathcal{C}}$, $\bar{T}_f x = \int f d\mu^x$. ($x \in$ domain $\bar{T}_f$ if and only if the right-hand integral exists.)*

For $y \in H$, let $\mu^{x,y}(U) = (\mu^x(U), y) = (\mu(U)x, y)$. Then $\mu^{x,y}$ is a finite, complex valued "measure," and $(\bar{T}_f x, y) = \int f d\mu^{x,y}$. The definition of integral coincides, for $\mu^{x,y}$, with the conventional one. However, 4.3 is stronger than the latter result—the sums approximating $\int f d\mu^x$ converge to $\bar{T}_f x$ in the norm topology of $H$, as well as in the weak.

For $f \in \bar{\mathfrak{C}}$ we define a spectral measure on the Borel subsets of the complex plane by $\nu(U) = \mu(f^{-1}(U))$. It is easy to check that for $U$ open $\nu(U)$ is the strong limit of the measures of compact subsets of $U$, so that $\nu$ is regular. Moreover, $\int f(p)d\mu p = \int Z d\nu Z$, for one may verify that the approximating sums are identical. Hence, $T_f = \int Z d\nu Z$ for $f \in \mathfrak{C}$, and $\bar{T}_f x = \int Z d\nu^x Z$ for $f \in \bar{\mathfrak{C}}$, where $\nu^x(U) = \nu(U)(x)$.

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† It is precisely this point at which the requirement that $Y$ be metric, seems essential. If $Y$ is not metric, but compact Hausdorff, a modification of the argument given here shows that $f = g$ save on a set $S$ with the property: if $\rho$ is any pseudometric on $X$, then $S$ is of first category with respect to the topology given by $\rho$.

[1] Stone, M. H., "A General Theory of Spectra I," PROC. NATL. ACAD. SCI., **26**, 280–283 (1940).

[2] Yosida, K., "Normed Rings and Spectral Theorems," *Proc. Imp. Acad. Tokyo*, **19**, 356–359 (1943).

[3] Stone, M. H., "Boundedness Properties in Function Lattices," *Can. J. Math.*, **1**, 176–186 (1949).

[4] Bochner, S., and Ky Fan, "Distributive Order-Preserving Operations in Partially Ordered Vector Sets," *Ann. Math.*, **48** (2), 168–179 (1947).

[5] Halmos, P. R., "Commutativity and Spectral Properties of Normal Operators," *Acta Sci. Math. Szeged*, **12**, 153–156 (1950).

[6] Gelfand, I. M., and Neumark, M. A., "On the Embedding of Normed Rings into the Ring of Operators in Hilbert Space," *Rec. Math. Moscou*, **12**, 197–213 (1943).

# *COMMUTATIVE OPERATOR ALGEBRAS*

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This is an investigation of the structure of a commutative strongly closed, self adjoint algebra $\mathfrak{A}$, with $1 \in \mathfrak{A}$, of operators on a Hilbert space $H$. For brevity, the paper is written as a continuation of the preceding note[1] assuming its notation and results. The principal theorem is: There is a unique multiplicity function $\phi$ on the spectrum $X$ of $\mathfrak{A}$ to the cardinal numbers, $\phi$ being continuous relative to the order topology for the cardinals; from $X$ and $\phi$ the space $H$ and the algebra $\mathfrak{A}$ may be reconstructed, to a unitary equivalence. (The intuitive motivation: If $x \in H$ is, for each $A \in \mathfrak{A}$, a characteristic vector with characteristic value $\lambda(A)$, then $\lambda$ is a homomorphism of $\mathfrak{A}$ into the complex numbers; i.e., $\lambda \in X$. Thus $X$ can be interpreted as the space of characteristic values, to each of which we