years (private communication), that the solutions given in Table 2 are the only possible solutions in positive integer values of n and w ($n \geq w$) of equation (8). Dr. W. Ljunggren (private communication) has independently confirmed Dr. Skolem's statement. The three lower pairs of values given in Table 2 satisfy equation (8) and are given in parentheses for completeness, but are not in agreement with Gaunt's condition $n \geq 2$. When $n \geq 2$, Table 2 represents *supplementary selection rules to those found by Gaunt* for integral (7).

I am indebted to Drs. Th. Skolem and W. Ljunggren, of the Institutes of Mathematics of the Universities of Oslo and Bergen, respectively, for solving the number-theory problems in this discussion and to Mr. Oddmund Kolberg for suggesting the transformation of equation (8) into equation (9) that made it possible to use standard methods for solving equation (8).

[1] J. A. Gaunt, *Phil. Trans. Roy. Soc. London*, **A228**, 151, 1929.

[2] L. Infeld and T. E. Hull, *Rev. Modern Phys.*, **23**, 21, 1951. Refer, in particular, to sec. 9, p. 51.

[3] J. Bird, Master's thesis, University of Toronto, 1949.

ON THE CONCEPT OF FIBER SPACE

BY WITOLD HUREWICZ*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Communicated by O. Zariski, August 16, 1955

The concept of the fiber space can be introduced either in terms of local properties or in terms of properties in the large. The purpose of this note is to clarify the relations between these two concepts.

Throughout this paper the word "space" means Hausdorff space. The word "map" means continuous map. The symbol I will denote the unit interval of real members. Given a space X, X^I is the space of parametrized paths in X with the usual c.o. topology.

1. FIBER SPACES AND LIFTING FUNCTIONS

Let p be a map of a space E into a space B. Let Ω_p be a subset of the Cartesian product $E \times B^I$ consisting of all pairs (e, ω) with

$$e \in E, \qquad \omega \in B^I, \qquad \omega(0) = p(e).$$

Let $\tilde{p}$ be the map of E^I into Ω_p which assigns to each $\tau \in E^I$ the pair $(\tau(0), p\tau)$.

Definition: The triple (E, B, p) is called a *fiber space* if the map $\tilde{p}$ has a cross-section, i.e., if there exists a map $\lambda: \Omega_p \to E^I$ such that $\tilde{p}\lambda$ is the identity map $\Omega_p \to \Omega_p$.

A map λ satisfying the condition above will be called a *lifting function* (belonging to p). Explicitly, λ assigns to each point $e \in E$, and each path $\omega \in B^I$ starting at $p(e)$, a path $\lambda(e, \omega)$ in E which starts at e and corresponds to ω under p.

While, generally speaking, the lifting function λ is not uniquely determined by the map p, it can easily be shown that all lifting functions (if any) form a single

homotopy class, i.e., any two distinct lifting functions can be connected by a homotopy within the family of lifting functions (for each number s, $0 \leq s \leq 1$, one constructs a lifting function λ_s by decomposing each path $\omega \in B^I$ into two arcs according to the partition of the parameter interval I into two subintervals of lengths $1 - s$ and s, respectively, and using λ_0 to lift the first arc and then λ_1 to lift the second arc).

A classical example of a fiber space is the triple (B^I, B, p), where B is an arbitrary space and p assigns to each path $\omega \in B^I$ its origin $\omega(0)$.

2. COVERING HOMOTOPY CONDITION

THEOREM. *In order that the triple (E, B, p) be a fiber space, it is necessary and sufficient that the "covering homotopy condition" be satisfied. This means that, given any space X, any map F: $X \times I \rightarrow B$, and a map f: $X \times 0 \rightarrow E$ such that pf coincides with F on $X \times 0$, f can always be extended to a map $\tilde{F}$: $X \times I \rightarrow E$, with $p\tilde{F} = F$.*[1]

Proof: Necessity: For each $x \in X$, denote by ω_x the path determined by $F(x, t)$, $0 \leq t \leq 1$, and define $\tilde{F}$ by mapping $x \times I$ onto the path $\lambda(f(x), \omega_x)$, where λ is a lifting function belonging to p. Sufficiency: Setting $X = \Omega_p$, $F(x, t) = F(e, \omega, t) = \omega(t)$, $f(x) = f(e, \omega) = e$, we take for $\lambda(x)$ the path $\tilde{F}(x, t)$, $0 \leq t \leq 1$.[2]

3. REGULAR FIBER SPACES

Treating B and E as subspaces of B^I and E^I, respectively, we call a lifting function λ *regular* if for every $e \in E$, $\lambda(e, pe) = e$ (in other words, "degenerated" paths in B consisting of a single point are lifted into degenerated paths in E). A triple (E, B, p) is called a *regular fiber space* if it admits a regular lifting function. This is equivalent to the condition that the covering homotopy theorem should be valid, with the additional requirement that those points of X (see notations in sec. 2) which are stable under the homotopy F (this means that $F(x, t)$ is constant as a function of t) should be stable under the homotopy $\tilde{F}$ as well. A covering homotopy condition fulfilling this requirement will be referred to as a "strong covering homotopy condition."

Not every fiber space is regular. For example, the fiber space of paths over B defined at the end of section 1 is not always regular. Specifically, it can be shown that the fiber space just mentioned fails to be regular if B is the joint of two enumerable (infinite), connected Hausdorff spaces. However, the condition of regularity is no restriction, if B is a *metric* space.

THEOREM. *Every fiber space (E, B, p), where B is a metric space, is regular.*

Proof: We may assume that the diameter of B is ≤ 1. Given $\omega \in B$,^{I} we denote by d_ω the diameter of the set $\omega(I)$ and define a "modified" path ω' by the relations

$$\omega'(t) = \omega\left(\frac{t}{d_\omega}\right) \qquad \text{if } t < d_\omega,$$

$$\omega'(t) = \omega(1) \text{ if } t \geq d_\omega.$$

Given a lifting function λ, a regular lifting function λ' can be obtained as follows:

$$\lambda'(e, \omega)\ (t) = \lambda(e, \omega')\ (d_\omega t), \qquad (e, \omega) \in \Omega_p, \qquad 0 \leq t \leq 1.$$

4. LOCAL FIBER SPACES

Let p be a map: $E \to B$. We shall call the triple (E, p, B) a *local fiber space* if for every point $b \in B$ there exists a neighborhood U of b such that the triple $(p^{-1}(U), U, p)$ (with p restricted to $p^{-1}(U)$) is a fiber space.

In a similar way we localize the concept of the regular fiber space.

Every fiber space is obviously a local fiber space. Remarkably, the converse statement holds under very general conditions. The main result of this paper is the following:

UNIFORMIZATION THEOREM. *A (regular) local fiber space (E, B, p) is a (regular) fiber space in the large, provided that B is paracompact (e.g., if B is metric).*

The theorem asserts that local lifting functions can be "matched" into a uniform global lifting function.

5. PROOF OF THE UNIFORMIZATION THEOREM

An open covering $\{U_r\}$ of a space X will be called *normal* if for every set U_r there exists a real-valued, continuous function defined on X, which has positive values on U_r and vanishes outside U_r. If X is a normal space, every (open) loc. finite covering of X has a refinement which is both loc. finite and normal. Using the fact that paracompact spaces are normal, it follows that every covering of a paracompact space has a normal loc. finite refinement. Hence the uniformization theorem is contained in the following statement, which does not depend on the assumption that B is paracompact.

5.1. *Let p be a map: $E \to B$. Suppose that B admits a loc. finite, normal covering $\{U_r\}$ such that for every U_r the triple $(p^{-1}(U_r), U_r, p)$ is a (regular) fiber space. Then the triple (E, B, p) is a (regular) fiber space.*

Given a subset W of B^1, let $\tilde{W}$ be the set of all triples (e, ω, s) with $e \in E$, $\omega \in W$, $0 \leq s \leq 1$, $\omega(s) = p(e)$. By an *extended lifting function* over W is meant a map Λ: $\tilde{W} \to E^1$, satisfying the condition

$$p\Lambda(e, \omega, s) = \omega, \ \Lambda(e, \omega, s)\ (s) = e$$

for every triple in $\tilde{W}$. The set W will be called *liftable* if an extended lifting function can be defined over W. It is quite easy to show that the whole path space B^I is liftable if and only if (E, B, p) is a fiber space. (In other words, the existence of a lifting function over B^I implies the existence of an extended lifting function.)[2a]

According to the assumptions of 5.1, each set U_r^I (regarded as a subset of B^I) is liftable. For each finite sequence of indices $(r_1, r_2, \ldots, r_k)$ we denote by $W_{r_1r_2\ldots r_k}$ the subset of B^I consisting of paths ω satisfying

$$\omega(t) \in U_{r_i} \quad \text{for } \frac{i-1}{k} \leq t \leq \frac{i}{k}, \qquad i = 1, 2, \ldots, k.$$

5.2. *The sets $W_{r_1r_2\ldots r_k}$ (with varying k) form a normal covering of B^I. This covering has a loc. finite normal refinement.*

The easy proof is based on the fact that for a fixed k the sets of 5.2 form a loc. finite system.

Now we show

5.3. *Each of the sets $W_{r_1r_2\ldots r_k}$ is liftable.*

Proof: Let Λ_i be an extended lifting function over $U_{\tau_i}{}^I$, $i = 1, 2, \ldots, k$. Consider a fixed triple $(e, \omega, s) \in \tilde{W}_{\tau_1\tau_2} \ldots {}_{\tau_k}$. Suppose that $(n - 1)/k \leq s \leq n/k$ (n an integer). For $i = 1, 2, \ldots, k$, let $\omega_i \in B^I$ denote the function which coincides with ω for $(i - 1)/k \leq t \leq i/k$ and is constant elsewhere. Now we define the path $\tau \in E^I$ in the following way. First, we define τ for the interval $(n - 1)/k \leq t \leq n/k$ by setting

$$\tau(t) = \Lambda_n(e, \omega_n, s)\ (t), \qquad \frac{n-1}{k} \leq t \leq \frac{n}{k},$$

and then set:

$$\tau(t) = \Lambda_{n-1}\left(\tau\left(\frac{n-1}{k}\right), \omega_{n-1}, \frac{n-1}{k}\right)(t), \qquad \frac{n-2}{k} \leq t \leq \frac{n-1}{k},$$

$$\tau(t) = \Lambda_{n+1}\left(\tau\left(\frac{n}{k}\right), \omega_{n+1}, \frac{n}{k}\right)(t), \qquad \frac{n}{k} \leq t \leq \frac{n+1}{k},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

We have $p\tau = \omega$ and $\tau(s) = e$. Consequently, the function which assigns τ to the triple (e, ω, s) is an extended lifting function. According to 5.2 and 5.3, the proof of 5.1 is completed if we prove the following lemma:

5.4. *Suppose that B^I admits a loc. finite, normal covering $\{W_\mu\}$ such that each set W_μ is liftable. Then the triple (E, B, p) is a fiber space.*

Proof of 5.4: For each set W_μ we select (a) a continuous, real-valued function $f_\mu(\omega)$ ($\omega \in B^I$) which has positive values on W_μ and vanishes outside W_μ, and (b) an extended lifting function Λ_μ over W_μ. We may assume that the set of indices μ is linearly ordered. For a given $\omega \in B^I$, let

$$\mu_1 < \mu_2 < \ldots < \mu_k$$

be all those indices for which $\omega \in W_\mu$. We set

$$q_r = \frac{\sum_{i=1}^{r} f_{\mu_i}(\omega)}{\sum_{i=1}^{k} f_{\mu_i}(\omega)}, \qquad r = 1, 2, \ldots, k.$$

Letting $e \in E$ be a point with $p(e) = \omega(0)$, define a path $\tau \in E^I$ (depending on ω and e) in the following way:

$$\tau(t) = \Lambda_{\mu_1}(e, \omega, 0)\ (t) \text{ for } 0 \leq t \leq q_1,$$
$$\tau(t) = \Lambda_{\mu_{i+1}}(\tau(q_i), \omega, q_i)\ (t) \qquad \text{for } q_i \leq t \leq q_{i+1}, \qquad i = 1, 2 \ldots k - 1.$$

The path τ depends continuously on (e, ω) and satisfies $\tau(0) = e$, $p\tau = \omega$. Hence we have defined a lifting function over B^I.

Remark: If local regularity is assumed in 5.1, an obvious modification of the preceding proof (regularity condition imposed on all lifting functions used in the proof) would yield a regular lifting function as an end result.

6. GENERALIZATIONS

Consider two triples $\alpha = (E, B, p)$ and $\alpha' = (E', B, p')$, where p and p' are maps of E and E', respectively, into B. Let $E'' \leq E \times E'$ consist of all pairs (e, e') with $p(e) = p'(e')$, and let p'' be the projection $E'' \to E'$. We shall say that α is a (regular) fiber space relative to α' if the triple (E'', E', p'') is a (regular) fiber space.

It is quite easy to see that an absolute (regular) fiber space is a relative (regular) fiber space with respect to every triple $\alpha' = (E', B, p')$. Furthermore, the triple $\alpha = (E, B, p)$ is an absolute (regular) fiber space if and only if α is a relative (regular) fiber space with respect to the triple (B, B, i), where i is the identity map.

The Uniformization Theorem implies

6.1. *A local (regular) fiber space (E, B, p) is a global fiber space relative to every triple (E', B, p') in which E' is paracompact.*

COROLLARY. *Any local fiber space (E, B, p) satisfies the covering homotopy condition for maps of paracompact spaces.*[3]

In fact, let X be paracompact and let $F: X \times I \to B$, $f: X \to E$, $pf = F|X \times 0$. Since, as well known, $X \times I$ is paracompact, (E, B, p) is a fiber space relative to the triple $(X \times I, B, F)$. Let $f': X \to E'' \leq X \times I \times E$ be defined by $f'(x) = (x, 0, f(x))$. We now apply the covering homotopy condition to the fiber space $(E'', X \times I, p'')$, substituting f' for f and the identity map $X \times I \to X \times I$ for F. This gives a map of $X \times I$ into E'', which, followed by the projection $E'' \to E$, yields the desired covering homotopy $\tilde{F}$.

Let us observe that in case (E, B, p) is locally a regular fiber space, F can be chosen so that the *strong* covering homotopy condition (see sec. 3) is fulfilled.

7. SLICING FUNCTIONS

Let p be again a map $E \to B$. Let V be a subset of $E \times B$. By a *slicing function* over V (belonging to the triple (E, B, p)) is meant a map σ: $V \to E$ such that (1) $p\sigma(e, b) = b$ for every $(e, b) \in V$ and (2) $(e, p(e)) \in V$ implies $\sigma(e, p(e)) = e$.

Consider the following properties of the triple (E, B, p):

1. There exists a neighborhood U of the diagonal D in $B \times B$ such that a slicing function can be defined over the set V consisting of pairs (e, b) with $(p(e), b) \in U$.

2. Each point b of B has a neighborhood W_b such that a slicing function can be defined over $p^{-1}(W_b) \times W_b$.

3. The triple (E, B, p) is a regular fiber space.

Property 1 essentially defines fiber spaces in the sense of Steenrod-Hurewicz.[4] Property 2 defines fiber spaces in the sense of Hu.[5] It is clear that property 1 implies property 2. It is equally clear that a triple satisfying property 2 is a local, regular fiber space. Therefore, according to the Uniformization Theorem, property 2 implies property 3, provided that B is paracompact. This means

7.1. *A triple with property 2 satisfies the strong covering homotopy condition for maps of arbitrary spaces. Without the assumption that B is paracompact, it still follows from* 6.1 *that the covering homotopy condition is satisfied for maps of paracompact spaces.*[6]

7.2. *If B is a metric absolute neighborhood retract, properties* 1, 2, *and* 3 *are equivalent.*

It suffices to show that property 3 implies property 1. As is well known, for a suitably chosen neighborhood U of the diagonal in $B \times B$ a map φ: $U \to B^I$ can be

defined such that, for $(b_1, b_2) \in U$, $\varphi(b_1, b_2)$ is a path joining b_1 to b_2 which degenerates into a single point in case $b_1 = b_2$. For $(p(e), b) \in U$, let $\sigma(e, b)$ denote the end point of the path $\lambda(e, \varphi(p(e), b))$, where λ is a regular lifting function. σ is clearly a slicing function. Incidentally, this shows that the set U in property 1 can be chosen independently of E and p.

* The results of this paper were presented to a seminar at the Institute for Advanced Study in January, 1954.

[1] This shows that fiber spaces in our sense are less general than fiber spaces in Serre's sense (see J. P. Serre, *Ann. Math.*, **54,** 425–505, 1951), who requires the covering homotopy condition only for maps of polyhedra. It is easy to give examples of fiber spaces in the sense of Serre which are not fiber spaces in the sense used in this paper.

[2] If E and B are metric spaces, then Ω_p is metric too, and it follows from the proof given above that in this case the covering homotopy property for maps of metric spaces implies the covering homotopy property for maps of arbitrary spaces.

[2a] For $\omega \in B^I$, $s \in I$, let ω_s, ω^s be paths defined by $\omega_s(t) = \omega(s - t)$ for $o \leqq t \leqq s$, $\omega_s(t) = \omega(o)$ for $s \leqq t \leqq 1$, $\omega^s(t) = \omega(s + t)$ for $o \leqq t \leqq 1 - s$, $\omega^s(t) = \omega(1)$ for $1 - s \leqq t \leqq 1$. Then, given a lifting function $\lambda(e, \omega)$, an extended lifting function $\Lambda(e, \omega, s)$ can be defined by setting $\Lambda(e, \omega, s)(t) = \lambda(e, \omega)(s - t)$ for $o \leqq t \leqq s$ and $\Lambda(e, \omega, s)(t) = \lambda(e, \omega)(t - s)$ for $s \leqq t \leqq 1$.

[3] If B is paracompact, then, according to the Uniformization Theorem, the covering homotopy condition is satisfied for maps of arbitrary spaces.

[4] W. Hurewicz and N. E. Steenrod, these PROCEEDINGS, **27,** 60–64, 1941.

[5] S. T. Hu, *Proc. Am. Math. Soc.*, **1,** 756–762, 1950.

[6] This result has been established independently by W. Huebsch in a recent paper (*Ann. Math.*, **61,** 555–563, 1955).

ON THE SPECTRAL SEQUENCE OF A FIBER SPACE

BY WITOLD HUREWICZ AND EDWARD FADELL

Massachusetts Institute of Technology and University of Wisconsin

Communicated by O. Zariski, August 16, 1955

1. *Introduction.*—Let (X, B, π) denote a regular fiber space in the sense of the preceding note,[1] let B be arcwise connected, and let $F = \pi^{-1}(b)$ be the fiber over a fixed element $b \in B$. Also, let $E_1, E_2, \ldots$ denote the spectral sequence associated with (X, B, π) and $d_1, d_2, \ldots$ the corresponding differential operators, with d_i: $E_i \to E_i$, $d_i^2 = 0$.[2] It is well known that E_1 is determined by B and F. Moreover, in case $\pi_1(B) = 0$ (or, more generally, in case $\pi_1(B)$ operates trivially on $H(F)$), E_2 is completely determined by B and F, and, in fact, $E_2^{p,q} = H_p(B, H_q(F))$. In the present note we outline a proof of the following theorem which generalizes this result. The details will appear elsewhere.

THEOREM. *If B is r-connected, i.e., $\pi_i(B) = 0$, $i \leq r$, then the first $r + 1$ terms of the spectral sequence $E_1, E_2, \ldots$ and the first r of the corresponding differential operators $d_1, d_2, \ldots$ are determined entirely by B and F. Hence $E_i = E_2$ for $2 \leq i \leq r + 1$ and $d_i = 0$ for $2 \leq i \leq r$ (since, as is well known, $E_i = E_2$ and $d_i = 0$ for $i \geq 2$ in case $X = B \times F$).*

2. *The Basic Map.*—Let u denote a singular cube in X of dimension n. A coordinate index i, $1 \leq i \leq n$, is called "db" (degenerate base) if