Thus both Dt and Mp are nonlinear in their respective actions, but the dosage effects are of opposite direction.

Summary.—The effect of Modulator on stability of the variegated pericarp allele (P^{vv} or $P^{rr}Mp$) in somatic tissue was studied in three otherwise near-isogenic genotypes designated "medium variegated" ($P^{rr}Mp/P^{wr}$), "light variegated" ($P^{rr}Mp/P^{wr}$ + tr $Mp/-$), and "homozygous variegated" ($P^{rr}Mp/P^{rr}Mp$). As Emerson had earlier observed in unrelated stocks, the single variegated allele in plants heterozygous for a stable P gene was found to mutate to self color approximately three times as frequently as either of the alleles in variegated homozygotes. It was postulated that the lower rate in the homozygote was the result of the action of the Modulator component in each variegated allele on mutation to self color of the other variegated allele. The test of this hypothesis which was applied was a comparison of the frequency of mutation in homozygous variegateds, carrying two P^{vv} alleles embodying two Modulator units, and light variegateds, carrying one variegated allele and one transposed Modulator. If the mutation rate of a P^{vv} allele is equally influenced by an additional Mp in any position, including the locus of the homologous allele, then twice as many mutations to self color would be expected in the homozygotes as in the light variegateds. The numerical results obtained on scoring the two classes of ears in question for mutations to the self color agreed closely with expectation on this hypothesis.

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ON MAUTNER'S EIGENFUNCTION EXPANSION

BY WILLIAM G. BADE AND JACOB T. SCHWARTZ

UNIVERSITY OF CALIFORNIA AND YALE UNIVERSITY

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The present note had its origin in a paper of Mautner[1] on the general eigenfunction expansion problem for self-adjoint operators on a Lebesgue space and recent papers of Gårding[2] and Browder[3] which apply Mautner's results to elliptic differential operators.

Let $(S, \sum, \nu)$ be a σ-finite measure space and T be a (possibly unbounded) self-adjoint operator in $L_2(S, \sum, \nu)$. A version of the spectral theorem asserts that there exist vectors $\{f_\alpha\}$ such that $L_2(S, \sum, \nu)$ is the orthogonal direct sum $\sum H_\alpha$, where H_α is the manifold of all vectors $F(T)f_\alpha$, where $F \epsilon L_2(\mu_\alpha)$, $\mu_\alpha = (E(\cdot)f_\alpha, f_\alpha)$. Moreover, the correspondence $F(T)f_\alpha \to F(\cdot)$ establishes an isometric isomorphism U of H onto the direct sum $\sum L_2(\mu_\alpha)$ which diagonalizes T in the sense that $(UF(T)f)_\alpha(\lambda) = F(\lambda)\ (Uf)_\alpha(\lambda)$ for $f \epsilon D(F(T))$. Following Mautner, we say that T has an *eigenfunction expansion* if for any such diagonalizing decomposition of $L_2(S, \sum, \nu)$ there exist kernels $W_\alpha(s, \lambda)$ such that the corresponding operator U is determined by the equations

$$(Uf)_\alpha(\lambda) = \int_S f(s)\overline{W_\alpha(s, \lambda)\nu(ds)}, \tag{1}$$

the integral existing in the mean-square sense[4] in $L_2(\mu_\alpha)$, and the inversion formula

$$f(s) = \sum_\alpha \int_{-\infty}^{\infty} (Uf)_\alpha(\lambda)W_\alpha(s, \lambda)\mu_\alpha(d\lambda) \tag{2}$$

holds, the integrals existing in the mean-square sense[4] in $L_2(S, \sum, \nu)$ and the series converging in the topology of $L_2(S, \sum, \nu)$.

1. *The Main Theorem.*—Our main theorem establishes a sufficient condition that an operator T should possess an eigenfunction expansion. In later sections we discuss the relation of this result to those of Mautner and Gårding. More complete details will appear in a forthcoming book.[5] It should be noted that we assume that $(S, \sum, \nu)$ is σ-finite but do not assume that $L_2(S, \sum, \nu)$ is separable.

1.1 DEFINITION: A bounded linear operator A in $L_2(S, \sum, \nu)$ has property (α) if there exists a sequence $\{S_n\}$ of sets of finite measure covering S such that

$$\nu\text{-ess}\sup_{S_n} |(Af)\ (s)| < \infty, \qquad n = 1, 2, \ldots,$$

for each $f \epsilon L_2(S, \sum, \nu)$. The sequence $\{S_n\}$ of sets will be called a *covering sequence* for the operator A.

If A has property (α), the map $Af \to (Af|S_n)$ of $(Af)\ (\cdot)$ into its restriction to S_n is a closed and hence bounded linear map of $L_2(S, \sum, \nu)$ into $L_\infty(S_n, \sum, \nu)$, for $n = 1, 2, \ldots$. In fact, this is an equivalent formulation of property (α).

1.2 THEOREM. *Let T be a self-adjoint operator in $L_2(S, \sum, \nu)$ with the resolution of the identity $E(\cdot)$. If for each bounded Borel set e of the real line the projection $E(e)$ has property (α), then T has an eigenfunction expansion.*

1.3. LEMMA. *Under the hypothesis of Theorem 1.2, there exists a common covering sequence $\{S_n\}$ for all the projections $E(e)$, i.e., a sequence $\{S_n\}$ such that*

$$\nu\text{-ess}\sup_{S_n} |(E(e)f)\ (s)| < \infty, \qquad n = 1, 2, \ldots,$$

for every bounded Borel set e and $f \epsilon L_2(S, \sum, \nu)$.

Clearly, if $\{C_n\}$ is a suitable covering sequence for $E(e)$ and $e_1 \subseteq e$, it is a suitable covering sequence for $E(e_1)$. Now let θ be a finite positive measure on $(S, \sum)$ such that $\theta(e_0) = 0$ implies $\nu(e_0) = 0$. Let $e_n = [-n, n]$, and select sets $A_n \subseteq S$ such that

$$\nu\text{-ess}\sup_{A_n} |(E(e_n)f)\ (s)| < \infty, \qquad f \epsilon L_2(S, \sum, \nu),$$

while the θ-measure of the complement A_n' of A_n satisfies $\theta(A_n') < 2^{-(n+1)}$. If

$S_1 = \bigcap_{n=1}^{\infty} A_n$, then $\theta(S_1') < 1/2$ and $\nu\text{-ess}\sup_{S_1} |(E(e)f)\,(s)| < \infty$ for every bounded Borel set e and $f \in L_2(S, \sum, \nu)$. The construction of $S_2, S_3, \ldots$ proceeds by induction.

We now construct the kernels $W_\alpha(s, \lambda)$.

1.4. LEMMA. *Let $\{S_n\}$ be a covering sequence for S such that formula* (3) *holds for every bounded Borel set e and $f \in L_2(S, \sum, \nu)$. Let $g \in L_2(S, \sum, \nu)$, and let $\mu(\cdot) = (E(\cdot)g, g)$. Then there exists a function W defined on the Cartesian product of S and the real line P, measurable with respect to $\nu \times \mu$, such that, if e is any bounded Borel subset of R,*

(i) $\nu\text{-}ess \sup_{s \in S_n} \int_e |W(s, \lambda)|^2 \mu(d\lambda) < \infty, n = 1, 2, \ldots,$

(ii) $(E(e)F(T)g)\,(s) = \int_e W(s, \lambda)F(\lambda)\mu(d\lambda),\ F \in L_2(\mu).$

Let $e_n = [-n, n]$. It follows easily that the map A_n: $F(\cdot) \rightarrow F(T)g$ is a closed and hence bounded map of $L_2(e_n, \mu)$ into $L_\infty(S_n, \sum, \nu)$. Consequently A_n^* is a bounded map of $L_\infty^*(S_n, \sum, \nu)$ into $L_2(e_n, \mu)$. Now $L_1(S_n, \sum, \nu)$ is isometrically imbedded in $L_\infty^*(S_n, \sum, \nu)$, and the restriction B_n of A_n^* is a bounded map of $L_1(S_n, \sum, \nu)$ into $L_2(e_n, \mu)$. Since $L_2(e_n, \mu)$ is separable, it follows from a theorem of Dunford and Pettis[6] that there exists a $(\nu \times \mu)$-essentially unique $\nu \times \mu$-measurable complex-valued function $W^{(n)}$ defined on $S_n \times e_n$ such that

$$\text{(iii)} \quad (B_nf)\,(\lambda) = \int_{S_n} f(s)W^{(n)}(s, \lambda)\nu(ds),\ f \in L_1(S_n, \textstyle\sum, \nu), \qquad \lambda \in e_n,$$

and such that

$$\nu\text{-ess}\sup_{S_n} \int_{e_n} |W^{(n)}(s, \lambda)|^2\mu(d\lambda) = \|B_n\|^2 < \infty.$$

The function W is defined by the formula

$$W(s, \lambda) = W^{(n)}(s, \lambda), \qquad (s, \lambda) \in S_n \times e_n.$$

Formula (ii) is easily established, using Fubini's theorem. *Q.E.D.*

Now let f_α be a family of vectors determining a diagonalizing decomposition of $L_2(S, \sum, \nu)$, and associate with each f_α the corresponding kernel W_α determined by Lemma 1.4. It follows by relatively standard arguments (cf., e.g., Browder, *op. cit.*) that for each $f \in L_2(S, \sum, \nu)$ the integrals $\int_{S_n} f(s)\overline{W_\alpha(s, \lambda)}\nu(ds)$ and $\int_{-n}^{n} (Uf)_\alpha(\lambda)W_\alpha(s, \lambda)\mu_\alpha(d\lambda)$ exist for almost all λ and s, respectively. Moreover,

$$(Uf)_\alpha(\lambda) = \lim_{n \rightarrow \infty} \int_{S_n} f(s)\overline{W_\alpha(s, \lambda)}\nu(ds),\ f \in L_2(S, \textstyle\sum, \nu), \tag{4}$$

and

$$(P_\alpha f)\,(s) = \lim_{n \rightarrow \infty} \int_{-n}^{n} (Uf)_\alpha(\lambda)W_\alpha(s, \lambda)\mu_\alpha(d\lambda), \quad f \in L_2(S, \textstyle\sum, \nu), \tag{5}$$

where P_α is the perpendicular projection of $L_2(S, \sum, \nu)$ onto H_α, the limits being in the topology of $L_2(\mu_\alpha)$ and $L_2(S, \sum, \nu)$, respectively. Formula (2) follows from the fact that $f = \sum_\alpha P_\alpha f$.

1.5. COROLLARY. *A self-adjoint operator T in $L_2(S, \sum, \nu)$ has an eigenfunction expansion if there exists a sequence $\{S_n\}$ of sets of finite ν-measure covering S such that every f in $\bigcap_{m=1}^{\infty} D(T^m)$ is ν-essentially bounded on each of the sets S_n.*

This useful corollary follows from the observation that $E(e)L_2(S, \sum, \nu) \subseteq D(T^m)$ for every m when e is bounded.

Mautner[1] showed that if, for some λ, the resolvent $R(\lambda; T)$ of a self-adjoint operator T in $L_2(S, \sum, \nu)$ was an integral operator of Carleman type (i.e.,

$$(R(\lambda; T)f)\ (s) = \int_S K(s, t)f(t)\nu(dt),$$

where K is product-measurable and $K(s, \cdot) \in L_2(S, \sum, \nu)$ for almost all s), then T has an eigenfunction expansion. Since the range of $R(\lambda; T)$ is $D(T)$, the domain of T, it follows immediately from Mautner's hypothesis that for each $g = R(\lambda; T)f \in D(T)$ we have

$$|g(s)| \leq |f|\theta(s),$$

where $\theta(s) = [\int_S |K(s, t)|^2\nu(dt)]^{1/2}$. On taking $S_n = \{s|\theta(s) \leq n\}$, Mautner's result follows from Corollary 1.5. Gårding[2] generalized Mautner's theorem for separable $L_2(S, \sum, \nu)$ to the case that some function $F(T)$ of T is a Carleman operator, where $F(\lambda)$ does not vanish on $\sigma(T)$.[7] In the case that $\inf_{[-n, n]} |F(\lambda)| > 0$, $n = 1, 2, \ldots$, it is easily seen that Gårding's hypothesis implies that each of the projections $E([-n, n])$ has property (α). (Otherwise we obtain hypothesis (α) for all the projections $E(e)$, where $\inf_{\lambda \in e} |F(\lambda)| > 0$, the set $\{\lambda | F(\lambda) = 0\}$ playing the role of the point $\lambda = \infty$ in Theorem 1.2. A straightforward modification of Theorem 1.2 will then yield the more general result of Gårding.)

2. *Elliptic Operators.*—Let S be an open subset of real Euclidean n-space, and let τ be a formally symmetric elliptic differential operator with sufficiently differentiable coefficients. Then τ determines a symmetric operator $T_0(\tau)$ defined on the C^∞ functions whose supports are interior to D. If T is any self-adjoint extension of $T_0(\tau)$, Gårding and Browder prove that an appropriate power of the resolvent of T is a Carleman operator and hence that T has an eigenfunction expansion. The functions $W_\alpha(\cdot, \lambda)$ belong to C^n and $(\tau - \lambda)W_\alpha(\cdot, \lambda) = 0$. Their construction of the Carleman kernel is based on a result of Fritz John which establishes the existence of a local fundamental solution for τ. We shall now indicate how the theorems of Gårding and Browder may be proved simply via our Corollary 1.5, without the introduction of fundamental solutions. We shall say a function $f \in L_2(S, \sum, \nu)$ possesses the partial derivative $\partial f/\partial x^i$ *in the mean-square sense* if

$$\lim_{h \to 0} h^{-1} f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n) = \frac{\partial f}{\partial x^i}$$

exists in the topology of $L_2(S, \sum, \nu)$. A function f belongs to the class H_p if it has mean-square partial derivatives of all orders $\leq p$ in each interior subdomain of S. If $p \geq [n/2] + 1$, then, by a result of Sobolev,[8] each f in H_p is continuous. It is known[9] that if m is the order of τ, then $D(T_0(\tau)^*)$, the domain of the adjoint of $T_0(\tau)$, consists of all functions f in H_m such that $\tau f \in L_2(S, \sum, \nu)$, and $T_0(\tau)^*f = \tau f$. Moreover, if $(T_0(\tau)^*f, h) = (g, h)$ for all $h \in D(T_0(\tau))$ and if $g \in H_k$, then $f \in H_{k+m}$. Thus, if T is any self-adjoint extension of $T_0(\tau)$, the functions in $D(T^r)$ $(\subseteq D((T_0(\tau)^*)^r)$ are continuous if $mr \geq [n/2] + 1$. It follows from Corollary 1.5 that T has an eigenfunction expansion, for we may take for $\{S_n\}$ any sequence of compact sets whose union is S.

In a recent paper L. Hörmander[10] has shown that a certain class of nonelliptic partial differential operators τ with constant coefficients, called by him *operators of local type*, also have the property that each $f \in \bigcap_{n=1}^{\infty} D((T_0(\tau))^n)$ is continuous. It follows then, by Corollary 1.5, that any self-adjoint extension of $T_0(\tau)$ has an eigenfunction expansion; this was shown by Hörmander, using Gårding's method of proof.

3. *Carleman Operators.*—Our next theorem shows that when $L_2(S, \Sigma, \nu)$ is separable, the hypotheses for Theorem 1.2 are actually equivalent to the requirement of Gårding that an appropriate function of T should be a Carleman operator.

3.1. THEOREM. *Let T be a self-adjoint operator in a separable space $L_2(S, \Sigma, \nu)$, and let each projection $E(e)$, bounded, have property (α). Then there exists a bounded function F defined on the real line such that $\inf_{[-n, n]} |F(\lambda)| > 0$, $n = 1, 2, \ldots$, and $F(T)$ is an integral operator of Carleman type.*

This result follows from a simple characterization of Carleman operators given in the next lemma. Since it may have independent interest, we state it for a possibly nonseparable space.

3.2 LEMMA. *Let (S, Σ, ν) be a σ-finite measure space. A bounded operator in $L_2(S, \Sigma, \nu)$ is of Carleman type if and only if it has a separable range and has property (α).*

Proof: Suppose

$$(Af)\ (s) = \int_S K(s, t)f(t)\nu(dt), \qquad f \in L_2(S, \Sigma, \nu),$$

where K is product-measurable and $\theta(s) = \int_S |K(s, t)|^2\nu(dt) < \infty$ a.e. We have already noted such an operator has property (α). To see that A has a separable range, construct operators A_n with kernels K_n such that $|K_n(s, t)| \le |K(s, t)|$, $\lim_{n\to\infty} K_n(s, t) = K(s, t)$ $\nu \times \nu$-a.e., and $\iint |K_n(s, t)|^2\nu(ds)\nu(dt) < \infty$. Then each A_n is a compact operator and hence has a separable range. For $E \in \Sigma$, define $(P_E f)\ (s) = f(s)\chi_E(s)$. If $\sup_E \theta(s) < \infty$, we note $|P_E(A_n f - Af)| \to 0$, $f \in L_2(S, \Sigma, \nu)$. If $E_n = \{s | \theta(s) \le n\}$, then $|Af - P_{E_n}Af| \to 0$, $f \in L_2(S, \Sigma, \nu)$. From this it follows easily that A has a separable range.

A proof of the sufficiency may be patterned on the proof of Theorem 1.2. Since A has a separable range, we may find a sub σ-field $\overline{\Sigma}$ of Σ such that $AL_2(S, \Sigma, \nu) \subseteq L_2(S, \overline{\Sigma}, \nu) \subseteq L_2(S, \Sigma, \nu)$ and $L_1(S, \overline{\Sigma}, \nu)$ and $L_2(S, \overline{\Sigma}, \nu)$ are separable. The map C_n: $f \to (Af | S_n)$ of $L_2(S, \Sigma, \nu)$ into $L_\infty(S_n, \overline{\Sigma}, \nu)$ is closed and bounded. The restriction B_n of C_n^* to $L_1(S_n, \overline{\Sigma}, \nu)$ is a bounded map into $L_2(S, \Sigma, \nu)$ having a separable range. Hence, by the theorem of Dunford and Pettis cited above, there exists a $\nu \times \nu$-essentially unique measurable function $K^{(n)}$ defined on $S_n \times S$ such that

$$(B_n f)\ (t) = \int_{S_n} f(s)K\ (s, t)\nu(ds),$$

$$f \in L_1(S_n, \overline{\Sigma}, \nu), \qquad t \in S,$$

and such that

$$\nu\text{-ess}\sup_{s \in S_n} \int_S |K^{(n)}(s, t)|^2\nu(dt) = \|B_n\|^2 < \infty.$$

The proof may be completed as before, to show that

$$(Af)\ (s) = \int_S K(s, t)f(t)\nu(dt), \qquad f \in L_2(S, \sum, \nu),$$

where $K(s, t) = K^{(n)}(s, t)$, $(s, t) \in S_n \times S$. *Q.E.D.*

To prove Theorem 3.1, let each projection $E(e)$, e bounded, have property (α), and select a sequence $\{S_n\}$, with the properties of Lemma 1.3, covering S. There exist numbers M_n such that

$$\nu\text{-ess sup}\ |(E([-n, n])f)\ (s)| \leq M_n|f|, \qquad s \in S_n.$$

Define $F(\lambda) = \sum_{n=1}^{\infty} M_n^{-1}\, 2^{-n}\chi_{[-n,\,n]}(\lambda)$, $\chi_{[-n,\,n]}$ denoting the characteristic function of the interval $[-n, n]$. Then

$$(F(T)f)\ (s) = \frac{\sum_{n=1}^{\infty} (E([-n, n])f)\ (s)}{2^n M_n}, \qquad f \in L_2(S, \sum, \nu),$$

the series converging uniformly on each of the sets S_n. It follows that $F(T)$ has property (α), and thus $F(T)$ is a Carleman operator by Lemma 3.2.

4. *Linear Independence of the Kernels* W_α.—If $L_2(S, \sum, \nu)$ is separable, it is known that there exists a diagonalizing sequence $\{f_n\}$ for T, a finite Borel measure μ on the real line R, and a sequence $R = e_1 \supseteq e_2 \supseteq \ldots$ of Borel sets such that $(E(e)f_n, f_n) = \mu(e \cap e_n)$, $n = 1, 2, \ldots$. We shall call the corresponding decomposition an *ordered* decomposition. Let W_i, $i = 1, 2, \ldots$, denote the corresponding kernels whose existence is proved in Lemma 1.4. Then $W_i(s, \lambda)$ may be supposed to vanish for $\lambda \notin e_i$. Since W_i is $\nu \times \mu$-measurable, the function $W_i(\cdot, \lambda)$ is ν-measurable for μ-almost all $\lambda \in e_i$.

4.1. THEOREM. *The ν-measurable functions $W_1(\cdot, \lambda), \ldots, W_n(\cdot, \lambda)$ are linearly independent for μ-almost all λ in e_n.*

The proof proceeds by induction. Suppose that the functions $W_i(\cdot, \lambda)$, $i = 1, \ldots, p - 1$, are linearly independent for μ-almost all λ on e_{p-1}, while the set $W_i(\cdot, \lambda)$, $i = 1, \ldots, p$, is not linearly independent for μ-almost all λ on e_p. By a fairly complicated argument, it is shown there exists a *Borel* subset $\sigma \subseteq e_p$ of positive μ-measure and *Borel-measurable* functions α_i such that

$$W_p(\cdot, \lambda) = \sum_{i=1}^{p-1} \alpha_i(\lambda)W_i(\cdot, \lambda), \qquad \lambda \in \sigma,$$

the principal tool being a vector form of the well-known theorem of Lusin. By integrating this equation over appropriate sets (cf. Lemma 1.4 [ii]), we may contradict the orthogonality of the manifolds $H_1, \ldots, H_p$. Complete details will be published elsewhere.

We may define the *multiplicity* of a projection $E(e)$ in the resolution of the identity for T as the number of sets e_i such that $\mu(e \cap e_i) \neq 0$ in an ordered decomposition which diagonalizes T. The multiplicity of the identity will also be called the *multiplicity of T*. Let T be a self-adjoint extension of the symmetric operator $T_0(\tau)$ corresponding to an ordinary differential operator τ of order n on an interval I. Theorem 4.1 may be used to show that the multiplicity of T is $\leq n$. The kernels $W_i(\cdot, \lambda)$ belong to $C^n(I)$ and satisfy the boundary conditions at a fixed end point of I or, more generally, at an end point at which there is a purely discrete spectrum (in a suitable sense). As a consequence we obtain additional information as to the magnitude of the multiplicity of T: e.g., let τ be a real second-order operator on an interval. If the boundary conditions determining T include a boundary condition at a limit-circle end point, then T has multiplicity one.[11]

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ON THE DISTRIBUTION OF CERTAIN FUNCTIONALS OF MARKOFF CHAINS AND PROCESSES

BY D. A. DARLING AND A. J. F. SIEGERT*

UNIVERSITY OF CHICAGO AND NORTHWESTERN UNIVERSITY

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1. *Functionals of Markoff Chains.*—Let x_n, $n = 0, 1, \ldots$ be a Markoff chain taking on values in an abstract space Ω and having stationary transition probabilities $P_n(x, E) = Pr\{x_{n+k} \in E | x_k = x\}$. Let $P_0(x, E) = \chi_E(x)$, the characteristic function of the set E, and suppose $x_0 = x$, a constant. Let $V(y)$, $y \in \Omega$, be real valued and measurable with respect to $\mathfrak{F}$, the Borel field of sets over which the P_n's are probability measures. In this section we study the distribution of the functional

$$U_n = \sum_{j=0}^{n} V(x_j). \tag{1.1}$$

This distribution will be determined, in principle, if we know the function

$$R(x, E) = \sum_{n=0}^{\infty} E\{e^{i\xi U_n} | x_0 = x, x_n \in E\} P_n(x, E) z^n, \quad 0 \leq z < 1, \xi \text{ real}, \tag{1.2}$$