GENERAL SOLUTION FOR THE STRESSES IN A CURVED MEMBRANE

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Communicated by R. Courant, October 3, 1957

For Cauchy's equations of equilibrium for a free continuous medium,[1] viz.,

$$t^{km}{}_{,m} = 0, \qquad t^{km} = t^{mk}, \tag{1}$$

where the comma denotes the covariant derivative, the general tensorial solution for a flat space of n dimensions has been known for some time.[2] For a curved space, such as an ordinary two-dimensional surface, a solution in tensorial form is not known, though various special results are available.[3]

In a penetrating memoir, L. Finzi[4] has exploited the long recognized similarity between the forms of certain known general solutions of equations (1) and the forms of the conditions of compatibility for the same space. He has shown that if the general solution is known, proper application of the converse of the principle of virtual work[5] yields the conditions of compatibility. Second, after assuming a certain form for the general solution of equations (1), he has determined the coefficients in that form explicitly by comparison with the result just mentioned, on the assumption that the conditions of compatibility are known. He has inferred that the solution so obtained is general and has illustrated it by exhibiting, for the first time, the general tensorial solution for a surface of revolution.

It does not detract from the originality and value of L. Finzi's work to point out that, while both his results are correct, the second inference rests on the general form assumed but not demonstrated for the solution of equations (1). Moreover, this second result may be obtained directly and in simpler form from the usual principle of virtual work. Indeed, that principle asserts that the tensor t^{km} satisfies equations (1) if and only if

$$\int t^{km} d_{km} dv = 0 \tag{2}$$

for all fields d_{km} such that $d_{km} = c_{(k,m)}$ for some vector c_k. Here and in what follows we systematically neglect boundary integrals as of no effect on the differential equations we seek.[6] This form of the principle, since it rests only on Green's theorem, is valid in any space where covariant differentiation is defined. Assume that the conditions of compatibility for the system $d_{km} = c_{(k,m)}$ have the explicit form

$$0 = \alpha^{km} d_{km} + \beta^{kmr} d_{km,r} + \gamma^{kmrs} d_{km,rs} + \delta^{kmrsp} d_{km,rsp} + \ldots, \tag{3}$$

where the tensors $\alpha^{km}, \beta^{kmr}, \ldots,$ are symmetric in the first two indices. Introducing a multiplier A, we may then replace (2) by

$$\int [t^{km} d_{km} - A(\alpha^{km} d_{km} + \beta^{kmr} d_{km,r} + \ldots)] dv = 0, \tag{4}$$

where the variation of d_{km} is now unrestricted. By Green's theorem follows the equivalent statement

$$\int [t^{km} - \alpha^{km}A + (\beta^{kmr}A)_{,r} - (\gamma^{kmrs}A)_{,rs} + \ldots]d_{km}dv = 0. \tag{5}$$

Hence

$$t^{km} = \alpha^{km}A - (\beta^{kmr}A)_{,r} + (\gamma^{kmrs}A)_{,rs} - (\delta^{kmrsp}A)_{,rsp} - \ldots. \tag{6}$$

We have shown, then, that *the general solution of (1) is of the form (6) whenever the conditions of compatibility have the form (3)*. This is equivalent to the result inferred by L. Finzi in the special case when the sum (3) terminates after the number of terms actually written down. The stress function A appears as the multiplier for the constraint expressing the compatibility of the deformation with a virtual displacement field for the space considered. It is perhaps necessary to emphasize that the above argument includes a *proof of completeness;* moreover, unlike the usual proofs based on the theorem of representation of a solenoidal field in terms of a vector potential, this proof is not merely local but, to the extent that the method of multipliers is valid in the calculus of variations, proves the existence of the stress functions *in the large.*

The form (3) of the conditions of compatibility is appropriate to spaces of two dimensions. For spaces of three dimensions, the tensors α^{km}, β^{kmn}, ..., are to be replaced by tensors α^{ijkm}, β^{ijkmr}, ... Accordingly, the scalar multiplier A in (4), and hence the stress function A in the solution (6), is to be replaced by a tensor A^{ij}.

Since the process applies equally when the coefficients in (3) are tensors of any order, knowledge of the conditions of compatibility is shown to be equivalent to the integration of (1), for any space where covariant differentiation is defined. There may, of course, be spaces where neither problem admits a solution.

To apply formula (6) to membranes,[7] we need the explicit form of the conditions of compatibility (3) in a Riemannian space of two dimensions with covariant metric tensor a_{km}. For a *surface of constant curvature* K, B. Finzi[8] has shown that

$$\begin{aligned} \alpha^{km} &= Ka^{km}, \qquad \beta^{kmr} = 0, \\ \gamma^{kmrs} &= -e^{s(k}e^{m)r}, \end{aligned} \tag{7}$$

where $\sqrt{a}e^{ks}$ is the permutation symbol such that $\sqrt{a}e^{12} = +1$ and where all coefficients of higher order vanish. Hence the general solution (6) for membranes of constant curvature is

$$t^{km} = e^{kp}e^{mq}A_{,pq} + KAa^{km}, \tag{8}$$

which is the result given by B. Finzi.[2] For a *surface applicable upon a surface of revolution,* B. Finzi[8] has shown that

$$\begin{aligned} \alpha^{km} &= K_{,p}K^{,p}a^{km} + K^{,k}K^{,m}, \\ \beta^{kmr} &= KK^{,r}a^{km} \qquad \gamma^{kmrs} = 0, \\ \delta^{kmrsp} &= -e^{s(k}e^{m)r}K^{,p}, \end{aligned} \tag{9}$$

while all coefficients of higher order vanish. Hence the general solution (6) for such a surface is

$$\begin{aligned} t^{km} = {} & [(K_{,p}K^{,p} - KK_{,p}{}^{,p})a^{km} + K^{,k}K^{,m} + e^{s(k}e^{m)r}K_{,rsp}{}^{,p}]A \\ & + [-KK^{,r}a^{km} + e^{s(k}e^{m)q}K_{,qs}{}^{,r} + (e^{r(k}e^{m)q} + e^{q(k}e^{m)r})K_{,qp}{}^{,p}]A_{,r} \\ & + [(e^{r(k}e^{m)q} + e^{q(k}e^{m)r})K_{,q}{}^{,s} + e^{s(k}e^{m)r}]A_{,rs} \\ & + e^{s(k}e^{m)r}K^{,p}A_{,rsp}, \end{aligned}$$

equivalent to the result inferred by L. Finzi.[4]

For a general surface not falling into the above-mentioned special classes, B. Finzi[8] has obtained a condition of compatibility consisting not in a single scalar equation but in a system of differential order 4 and tensorial order 2. Using this result and the method described in the paragraph starting after equation (6), for the stresses in a membrane of arbitrary form we could easily exhibit a general solution in terms of the fourth derivatives of a second-order tensor of stress functions. However, the result of Storchi[3] in geodesic co-ordinates indicates the existence of a solution in terms of fifth derivatives of a single stress function. Conversely, from this fact and the first result of L. Finzi[4] it follows that there exists a single scalar condition of compatibility, of fifth order, for an arbitrary curved surface. If that condition can be calculated, then the result of this paper will yield at once the general tensorial solution for the stresses in a membrane.

It should be possible to approach the problem of deriving conditions of compatibility for the system $d_{km} = c_{(k,\, m)}$ by a method valid in a general affine space. Apparently it is difficult; I have not been able to find any literature concerning it.

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1 Various mechanical interpretations are possible. In the simplest, t^{km} is the stress tensor in a medium in equilibrium subject to no body force. In other interpretations, t^{km} includes the momentum transfer stress of a steady motion and/or a stress equivalent to a conservative body force. In still others, t^{km} is a combined stress-momentum tensor of a not necessarily steady motion. For a medium subject to arbitrary load F^k, $(1)_1$ is to be replaced by $t^{km}{}_{,m} + F^k = 0$; to solve this equation when the general solution of $(1)_1$ is known, only a particular integral is required.

For the case of a two-dimensional surface imbedded in three-dimensional Euclidean space, the solution of (1) constitutes the classical membrane problem, on which there is a literature dating back to Beltrami. In this context the equilibrium of tangential forces and of moments is expressed by (1), while the equilibrium of normal forces is expressed by the linear algebraic equation

$$b_{km}t^{km} + F = 0, \tag{A}$$

where b_{km} is the second fundamental form and F is the normal pressure. When the general solution of (1) in terms of a stress function is put into eq. (A), this latter becomes a differential equation for the stress function.

Mention should be made of the solution given by Pucher (1934), the subject of a recent exposition by L. Finzi, *Rend. Ist. Lombardo sci.*, **88**, 907–916, 1955. In this solution, based on a special oblique co-ordinate system, eqs. (1) and (A) are combined at the start, so that the intrinsic problem is not treated.

Results based on (1) are not to be confused with those appropriate to a *shell*, which may sustain cross-forces, bending moments, and twisting moments, as well as surface stresses.

2 The result is due to R. F. Gwyther, *Mem. Proc. Manchester Lit. & Phil. Soc.*, Vol. **56**, No. 10, 1912; and B. Finzi, *Rend. accad. naz. Lincei*, **19**, 578–584, 620–623, 1934. An elegant general proof was recently given by W. S. Dorn and A. Schild, *Quart. Appl. Math.*, **14**, 209–213, 1956. There is a burgeoning literature on the subject.

3 The general solution in geodesic co-ordinates was derived by E. Storchi, *Rend. accad. naz. Lincei*, **8**, 116–120, 326–331, 1950. A tensorial solution for spaces of constant curvature had been asserted by B. Finzi, *loc. cit.*

4 *Rend. accad. naz. Lincei*, **20**, 205–211, 338–342, 1956.

5 In the form given, e.g., by Dorn and Schild, *loc. cit.*

6 The reader who doubts this may dispose of surface integrals rigorously by an argument parallel to that used in the cited work of Dorn and Schild.

7 It should be unnecessary to remark that restricting the virtual displacements in (2) to be vectors in the two-dimensional Riemannian space in no way restricts actual displacements such as the membrane may have undergone in order to assume the figure of equilibrium in which (1) is satisfied.

8 B. Finzi, *Rend. Ist. Lombardo sci.*, **63**, 975–982, 1930.