

**THE IDENTITY OF THE WEAK AND STRONG EXTENSIONS OF A
LINEAR ELLIPTIC DIFFERENTIAL OPERATOR: II**

BY M. S. NARASIMHAN

TATA INSTITUTE OF FUNDAMENTAL RESEARCH, BOMBAY, INDIA

Communicated by S. Bochner

1. In a previous note under the same title¹ we proved the identity of the weak and strong extensions of certain linear elliptic operators. In this note we prove the same result for the most general linear elliptic operator with C^∞ coefficients in an arbitrary open subset Ω of R^n . We use the same notations as in the previous paper.

We observe that, while the theorem given here includes both the theorems of the previous note, the method of proof adopted here does not yield a result which the method of proof of Theorem 2 of the previous note does (at least under certain restrictions) namely, that D_W is the closure of its restriction to analytic functions in its domain if the coefficients are analytic.

2. **THEOREM.** *Let Ω be an arbitrary open subset of R^n . Let D be an elliptic operator in the sense that*

$$p(x, \xi) \equiv \sum_{j_1 + \dots + j_n = m} a_{j_1 \dots j_n}(x) \xi_1^{j_1} \dots \xi_n^{j_n}$$

is different from zero for each $x \in \Omega$ and for every real nonzero vector (ξ_1, \dots, ξ_n) . Then the weak and strong extensions of D_1 coincide.

Proof:

Since D_W is a closed operator in L^2 , the domain M of D_W is a Hilbert space with the scalar product

$$((f_1, f_2)) = (f_1, f_2)_{L^2} + (Df_1, Df_2)_{L^2}, \quad f_1, f_2 \in M.$$

We have to show that $\mathcal{E} \cap M$ is dense in M in the topology of M , where \mathcal{E} denotes the space of C^∞ functions in Ω . Evidently the space \mathcal{D} of C^∞ functions with compact support in Ω is contained in M . Let H denote the closure of \mathcal{D} in M , in the topology of M . Then if H^\perp denotes the orthogonal complement of H in the Hilbert space M , we have

$$M = H \oplus H^\perp. \tag{1}$$

We shall show that the elements of H^\perp are C^∞ functions. Then it would follow from decomposition (1) and the fact the \mathcal{D} is dense in H (in the topology induced from M) that $\mathcal{E} \cap M$ is dense in M .

Now if $f \in H^\perp$, we have, for each $\varphi \in \mathcal{D}$,

$$(f, \varphi)_{L^2} + (Df, D\varphi)_{L^2} = 0;$$

i.e., f is a distribution solution of the equation $(D^*D + 1)f = 0$, where D^* denotes the formal adjoint of D . A simple computation shows that $(D^*D + 1)$ is an elliptic operator of order $2m$ (whose characteristic form is equal to $(-1)^m |p(x, \xi)|^2$). Since $(D^*D + 1)$ is a linear elliptic operator with C^∞ coefficients, every distribution solution f of the equation $(D^*D + 1)f = 0$ is indefinitely differentiable². Hence $H^\perp \subset \mathcal{E}$.

¹ These PROCEEDINGS, 43, 513, 1957.

² L. Gårding, *Math. Scand.*, 1, 55-72, 1953.