since (i) and (ii) are also valid for μ_0 and ν_0. This proves (iii).

As an application of Theorem 3, let μ be real. Then ν is real by (ii). Hence $\nu = 0$. Then $\mu = 0$ by (i). It follows that, for any continuous function h on B, there exists a sequence of polynomials whose real parts converge uniformly to h on B. By the maximum modulus principle, the sequence of real parts must converge uniformly on C, and the limit function will be harmonic on U. Thus we have:

THEOREM 4. *Let h be a continuous function on B. Then h can be extended to be continuous on C and harmonic on U, and the extended function can be uniformly approximated on C by real parts of polynomials.*

This theorem includes the solution of the Dirichlet problem for C, which is known. The further fact that the resulting harmonic extension of h can be uniformly approximated by real parts of polynomials seems to be new.

As an application of Theorem 4, one can prove the following theorem,[4] whose proof will be given elsewhere:

THEOREM 5. *If A is a uniformly closed algebra of continuous functions on B, which includes the polynomials, then either each function of A can be extended to be continuous on C and analytic on U, or A consists of all continuous functions on B.*

This generalizes a well-known result of Wermer.[5]

* This work was done with the support of ONR Contract NONR-222(37).

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NON-PARALLELIZABILITY OF THE n-SPHERE FOR $n > 7$

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We shall show that the recent results of R. Bott[1, 2]: $\pi_{2q}(U(q)) \approx Z/q!Z$ and the periodicity of the stable homotopy groups of $SO(n)$ and $U(n)$ imply the

THEOREM. *For $s \geqq 3$, the sphere S_{4s-1} of dimension $4s - 1$ is not parallelizable.*

Recall that a differentiable closed manifold of dimension n is said to be *parallelizable* if it admits a continuous field of tangent n-frames. It is well known that S_{4s+1}, $s \geqq 1$, is not parallelizable.[3, 7, 8] Thus S_1, S_3, and S_7, which are known to be parallelizable, are the only spheres which have this property.

Let α: $SO(n) \to U(n)$ and β: $U(n) \to SO(2n)$ be the standard injections. α sends a matrix $A \in SO(n)$ into itself, the entries of $\alpha(A)$ being regarded as complex numbers. β sends $C = (c_{u,v})$ into the $2n \times 2n$ matrix W, given by

$$w_{2u,2v} = w_{2u-1,2v-1} = a_{u,v}, \qquad w_{2u-1,2v} = -w_{2u,2v-1} = b_{u,v}$$

$(1 \leqq u, v \leqq n, c_{u,v} = a_{u,v} + ib_{u,v})$. We are interested in the induced homomorphisms on homotopy groups $\alpha_*:\pi_k(SO(n)) \to \pi_k(U(n))$, $\beta_*:\pi_k(U(n)) \to \pi_k(SO(2n))$, and more particularly in the composition $\beta_*\alpha_*$: $\pi_k(SO(n)) \to \pi_k(SO(2n))$.

LEMMA 1. *In the stable range, i.e., for $k < n - 1$, one has $\beta_*\alpha_* = 2i_*$, where i_*: $\pi_k(SO(n)) \to \pi_k(SO(2n))$ is induced by the inclusion i: $SO(n) \to SO(2n)$.*

Since by R. Bott,[1] $\pi_{4s-1}(SO)$ and $\pi_{4s-1}(U)$ are infinite cyclic, the homomorphisms α_*: $\pi_{4s-1}(SO(n)) \to \pi_{4s-1}(U(n))$ and β_*: $\pi_{4s-1}(U(n)) \to \pi_{4s-1}(SO(2n))$ are represented (in the stable range, $4s + 1 \leqq n$) by multiplication with integers a_s, resp. b_s (which are determined only up to sign, since we do not specify our choice for the generators of $\pi_{4s-1}(SO)$ and $\pi_{4s-1}(U)$). From the above lemma follows the

COROLLARY: $a_s \cdot b_s = 2$.

It is not difficult to obtain $a_1 = 2$, $a_2 = 1$.

LEMMA 3. *a_s and b_s are periodic of period 2, i.e., $a_{s+2} = a_s$, $b_{s+2} = b_s$.* (This information is not needed for the theorem to be proved).

Consider now the commutative diagram in which n is to be large ($2s < n$):

$$\begin{array}{ccccccc} \pi_{4s-1}(SO(2n)) & \xrightarrow{p_*} & \pi_{4s-1}(V_{2n,2n-4s+2}) & \to & \pi_{4s-2}(SO(4s-2)) & \to & 0 \\ \uparrow \beta_* & & \uparrow \beta'_* & & \uparrow & & \\ \pi_{4s-1}(U(n)) & \xrightarrow{q_*} & \pi_{4s-1}(W_{n,n-2s+1}) & \to & \pi_{4s-2}(U(2s-1)) & \to & 0, \end{array}$$

where the rows are portions of the homotopy sequences of the fibrations $SO(2n)/SO(4s-2) = V_{2n,2n-4s+2}$ with projection p, and $U(n)/U(2s-1) = W_{n,n-2s+1}$ with projection q, respectively. β': $U(n)/U(2s-1) \to SO(2n)/SO(4s-2)$ is induced by β: $U(n) \to SO(2n)$.

By R. Bott, $\pi_{4s-2}(U(2s-1)) \approx Z/(2s-1)!Z$. Hence q_* is (up to sign) the multiplication by $(2s-1)!$. By B. Eckmann,[5] section 3.6, $\pi_{4s-1}(V_{2n,2n-4s+2}) \approx Z_4$. For $s \geqq 3$, $(2s-1)!$ is divisible by 4, and thus $\beta'_*q_* = p_*\beta_* = 0$.

We need the following, probably well-known

LEMMA 2. *If S_{4s-1} is parallelizable, then $\pi_{4s-2}(SO(4s-2)) = 0$.*

Let us assume now that S_{4s-1} is parallelizable. Then p_*: $\pi_{4s-1}(SO(2n)) \to \pi_{4s-1}(V_{2n,2n-4s+2})$ is an epimorphism and hence maps a generator of $\pi_{4s-1}(SO(2n))$ into a generator of $\pi_{4s-1}(V_{2n,2n-4s+2})$. Therefore, $p_*\beta_*\epsilon_U = b_s \cdot \epsilon_V$, where ϵ_U, ϵ_V are generators of $\pi_{4s-1}(U(n))$, $\pi_{4s-1}(V_{2n,2n-4s+2})$, respectively. By Lemma 1, b_s is either 1 or 2 and hence $p_*\beta_* \neq 0$. Consequently, $s < 3$.

Proof of Lemma 1: Let A be the $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix},$$

i.e., $a_{ij} = \delta_{i+n,j}$ for $1 \leqq i \leqq n$ and $a_{ij} = \delta_{i-n,j}$ for $n < i \leqq 2n$. Since

$$i(X) = \begin{pmatrix} X & 0 \\ 0 & E \end{pmatrix},$$

we have

$$i(X) \cdot A \cdot i(X) \cdot A = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}.$$

Let M be the $2n \times 2n$ matrix given by

$$m_{ij} = \begin{cases} \delta_{2i-1,j} & \text{for} \quad 1 \leqq i \leqq n, \\ \delta_{2(i-n),j} & \text{for } n < i \leqq 2n. \end{cases}$$

It is easily verified that if $\boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{u}_2, \boldsymbol{v}_2, \ldots, \boldsymbol{u}_n, \boldsymbol{v}_n$ are the row vectors of a $2n \times 2n$ matrix Y, then the matrix $M \cdot Y$ has the row vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. This implies, using $(Y \cdot M')' = M \cdot Y'$ that if Y has column vectors $\boldsymbol{u}'_1, \boldsymbol{v}'_1, \boldsymbol{u}'_2, \boldsymbol{v}'_2, \ldots, \boldsymbol{u}'_n, \boldsymbol{v}'_n$, then $Y \cdot M'$ has the column vectors $\boldsymbol{u}'_1, \ldots, \boldsymbol{u}'_n, \boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_n$. From this it follows that

$$\beta\alpha(X) = M \cdot i(X) \cdot A \cdot i(X) \cdot A \cdot M'$$

for any matrix $X \in \boldsymbol{SO}(n)$. Notice that $|A| = (-1)^n$ and $|M| = (-1)^{n(n-1)/2}$. Since we are interested in the stable range $(k - 1 < n)$, there is no loss of generality in assuming n divisible by 4. Then $|A| = |M| = +1$. From the existence of paths from A and M to the unit $2n \times 2n$ matrix, follows: $\beta\alpha$ is homotopic to the map $\boldsymbol{SO}(n) \to \boldsymbol{SO}(2n)$, which sends X into $i(X) \cdot i(X) = i(X^2)$. By B. Eckmann,[4] Satz II, for any map $f: S_k \to \boldsymbol{SO}(n)$, the map f^2 given by $f^2(x) = [f(x)]^2$ represents $2\{f\}$, where $\{f\}$ is the homotopy class of f. This proves Lemma 1.

Proof of Lemma 2: By P. J. Hilton and J. H. C. Whitehead,[6] Lemma (4.12), if S_{4s-1} is parallelizable, then $\alpha_{4s-2} = \phi_* \mu$, where α_{4s-2} is the generator of $\pi_{4s-1}(S_{4s-2})$ the homomorphism ϕ_*: $\pi_{4s-1}(\boldsymbol{SO}(4s - 1)) \to \pi_{4s-1}(S_{4s-2})$ is induced by the projection ϕ: $\boldsymbol{SO}(4s - 1) \to S_{4s-2}$ and μ is some element in $\pi_{4s-1}(\boldsymbol{SO}(4s - 1))$. From the homotopy sequence of $\boldsymbol{SO}(4s - 1)/\boldsymbol{SO}(4s - 2) = S_{4s-2}$, i.e.,

$$\ldots \to \pi_{4s-1}(\boldsymbol{SO}(4s - 1)) \xrightarrow{\phi_*} \pi_{4s-1}(S_{4s-2}) \to$$
$$\pi_{4s-2}(\boldsymbol{SO}(4s - 2)) \to \pi_{4s-2}(\boldsymbol{SO}(4s - 1)) \to 0,$$

and the fact that ϕ_* is an epimorphism, it follows that

$$\pi_{4s-2}(\boldsymbol{SO}(4s - 2)) \approx \pi_{4s-2}(\boldsymbol{SO}(4s - 1))$$

(if S_{4s-1} is parallelizable).

Consider the homotopy sequence of $\boldsymbol{SO}(4s)/\boldsymbol{SO}(4s - 1) = S_{4s-1}$,

$$\ldots \to \pi_{4s-1}(\boldsymbol{SO}(4s)) \xrightarrow{\psi_*} \pi_{4s-1}(S_{4s-1}) \to \pi_{4s-2}(\boldsymbol{SO}(4s - 1)) \to \pi_{4s-2}(\boldsymbol{SO}(4_s)) = 0.$$

Since S_{4s-1} is assumed to be parallelizable, ψ_* is an epimorphism and therefore, $\pi_{4s-2}(\boldsymbol{SO}(4s - 1)) = 0$. This proves Lemma 2.

Proof of Lemma 3: By formula (3.4) of R. Bott,[1] the space of loops over $\boldsymbol{SO}$ has the same homotopy groups as the quotient space $\boldsymbol{SO}/\boldsymbol{U}$. Thus for $2s < n$,

$$\pi_{4s-1}(\Omega \boldsymbol{SO}(2n)) \approx \pi_{4s-1}(\boldsymbol{SO}(2n)/\boldsymbol{U}(n)).$$

However, $\pi_{4s-1}(\Omega \boldsymbol{SO}(2n)) \approx \pi_{4s}(\boldsymbol{SO}(2n))$, this latter group being 0 for odd s and $\mathbf{Z}_2$ for s even. Since the order of $\pi_{4s-1}(\boldsymbol{SO}(2n)/\boldsymbol{U}(n))$ is clearly equal to b_s, we obtain: $b_s = 1$ or 2 according as to whether s is odd or even, respectively. The equality $a_s \cdot b_s = 2$ yields the result for a_s, which could also have been obtained directly using (3.3) of Bott.[1]

LEMMA 4. *For s odd $\geqq 3$, the generator of $\pi_{4s-1}(S_{4s-2})$ does not belong to Im ϕ_*, where ϕ_*: $\pi_{4s-1}(\boldsymbol{SO}(4s - 1)) \to \pi_{4s-1}(S_{4s-2})$ is induced by the natural projection.*

Proof: We have seen that $s \geqq 3$ implies $p_*\beta_* = 0$. By Lemma 3, β_* is an epimorphism for s odd. Consequently, p_* must be trivial and $\pi_{4s-2}(\boldsymbol{SO}(4s - 2)) \approx Z_4$. The exact homotopy sequence of the fibration $\boldsymbol{SO}(4s - 1)/\boldsymbol{SO}(4s - 2) = S_{4s-2}$ then yields the result.

The original version of this paper did not contain Lemmas 3 and 4. Lemma 3 was also observed by the referee. I understand from R. Bott that J. Milnor has also obtained our theorem.

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