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A THEOREM ON FORCE-FREE MAGNETIC FIELDS

BY L. WOLTJER

UNIVERSITY OF CHICAGO

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Chandrasekhar and Woltjer[1] have shown that the force-free magnetic fields with "constant α" can be related to a variational principle. Here α is the scalar function in the equation

$$\text{curl } \boldsymbol{H} = \alpha \boldsymbol{H}, \tag{1}$$

which characterizes the force-free fields. This equation follows from the definition of force-free magnetic fields as fields in which the Lorentz force vanishes.

It was shown that the force-free fields with constant α are among the fields with maximum magnetic energy for a given mean square current density. Alternatively, this was formulated by stating that these fields are among the fields of minimum dissipation for a given magnetic energy.

There are several reasons why the above-mentioned formulations are not entirely satisfactory. It is not obvious physically why we should consider the mean square current density as given. Moreover, these formulations dc not lead us immediately to force-free fields but to a wider class of fields, which satisfy the equation

$$\text{curl curl } \boldsymbol{H} = \alpha^2 \boldsymbol{H}; \tag{2}$$

and additional arguments are then needed to arrive at equation (1). Difficulties also arise because of the presence of surface currents.

In this paper we shall prove another variational principle which states that the force-free fields with constant α represent the state of lowest magnetic energy in a closed system. A principle of this kind seems intuitively obvious, because, if the magnetic energy is a minimum, the field can produce no motions, and therefore the Lorentz force must vanish. The proof of the principle consists in two parts: We first show that in a closed system

$$\int_V \boldsymbol{A} \cdot \text{curl } \boldsymbol{A} dV = \text{constant}, \tag{3}$$

if the gauge of the vector potential $\boldsymbol{A}$ is suitably chosen. Next we show that, if we minimize the magnetic energy while satisfying equation (3), we obtain the force-free condition with a constant α.

If the conductivity is infinite, the induction equation is

$$\frac{\partial H}{\partial t} = \text{curl } (v \times H). \tag{4}$$

where v is the fluid velocity. Introducing the vector potential and choosing the gauge so that the scalar potential vanishes, this becomes

$$\frac{\partial A}{\partial t} = v \times \text{curl } A. \tag{5}$$

Therefore

$$\text{curl } A \cdot \frac{\partial A}{\partial t} = 0. \tag{6}$$

Making use of this relation, we obtain, on integrating by parts,

$$\frac{\partial}{\partial t}\int_V A \cdot \text{curl } A dV = \int_V A \cdot \text{curl } \frac{\partial A}{\partial t} dV$$
$$= \int_V \text{curl } A \cdot \frac{\partial A}{\partial t} dV + \int_S A \times \frac{\partial A}{\partial t} dS = 0. \tag{7}$$

The surface integral vanishes because we consider a closed system. For then the motions inside the system may not affect the vector potential outside, and, as the vector potential is continuous, even when surface currents are present, $\partial A/\partial t$ must vanish at the surface of the system.

We now ask for the minimum of the magnetic energy,

$$\frac{1}{8\pi}\int_V |\text{curl } A|^2 \, dV, \tag{8}$$

subject to condition (3). Making use of a Lagrangian multiplier α, we find that this requires

$$\int_V [2 \text{ curl } A \cdot \text{curl } \delta A - \alpha(\delta A \cdot \text{curl } A + A \cdot \text{curl } \delta A]dV = 0. \tag{9}$$

Integrating by parts and observing that δA must vanish at the surface for the reasons we have already given, we obtain

$$\int_V [\text{curl curl } A - \alpha \text{ curl } A] \cdot \delta A dV = 0. \tag{10}$$

Since δA is arbitrary, the integrand must vanish identically, and we must have

$$\text{curl curl } A - \alpha \text{ curl } A = 0. \tag{11}$$

Because of the definition of the vector potential, this is equivalent to

$$\text{curl } H = \alpha H. \tag{12}$$

Thus the force-free fields with constant α represent the lowest state of magnetic energy which a closed system may attain.

This has two important consequences. It proves in a general way the stability of force-free fields with constant α. Under restricted circumstances the stability of force-free fields with constant α has been demonstrated elsewhere more explicitly.[2] It also shows that in a system in which the magnetic forces are dominant and in which there is a mechanism to dissipate the fluid motions the force-free fields *with constant* α are the natural end configurations. The presence and stability of force-free fields when α is not a constant are not excluded by the present arguments, although they appear somewhat unlikely, as they are not stable against large perturbations.

Some mechanism is needed to dissipate the velocities. In astrophysical cases the viscosity is too small to be effective. But it seems not improbable that sufficient dissipation may be obtained by the Fermi mechanism of the acceleration of cosmic-ray particles. Similar suggestions have already been made for the dissipation of motions in the Crab nebula[3] and in interstellar space.[4] The present discussion is relevant if the conductivity is sufficiently large, i.e., if the time of decay of the magnetic field is longer than the damping time for the motions. This condition will probably be fulfilled in most astrophysical situations.

It is a pleasure to acknowledge many valuable discussions on force-free fields with Professor S. Chandrasekhar.

Note added in proof. Since this paper was written we have found that the hydromagnetic equations have other integrals besides the integral (3). The very general equilibrium configurations which may be derived from these integrals will be discussed in a forthcoming paper.

1 S. Chandrasekhar and L. Woltjer, "On Force-Free Magnetic Fields," these PROCEEDINGS, **44**, 285, 1958.

2 L. Woltjer, "The Stability of Force-Free Magnetic Fields," *Ap. J.* (in press).

3 L. Woltjer, "The Crab Nebula," *B.A.N.*, **14**, 39 (No. 483), 1958.

4 E. N. Parker, "Origin and Dynamics of Cosmic Radiation," *Phys. Rev.*, **109**, 1328, 1958.

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*THE EFFECTS OF HYDROXYLAMINE ON THE $C^{14}O_2$ FIXATION PATTERN DURING PHOTOSYNTHESIS**

BY J. A. BASSHAM, M. KIRK, AND M. CALVIN

RADIATION LABORATORY AND DEPARTMENT OF CHEMISTRY, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA

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Evidence for an unstable product or products formed in the early stages of carbon fixation during photosynthesis by algae has been presented by Metzner *et al.*[1] In an attempt to stabilize, isolate, and characterize such unstable products, Metzner *et al.*[2] added hydroxylamine to the algae just before, during, or after a short period of photosynthesis with $C^{14}O_2$.

Analysis of the products of such fixation by the usual techniques of paper chromatography and radioautography led to observation of an apparently new radioactive compound on the chromatogram. This compound or "spot" usually appeared in the two-dimensional papers as a thin streak (its long dimension in the phenol direction) somewhere between aspartic and malic acids. Observations that the spot