detailed treatment of it in Villat's Hydrodynamics.[7] If $G(r, t)$ is a solution of the heat equation,

$$\frac{\partial G}{\partial t} - \nu \frac{\partial^2 G}{\partial r^2} = 0, \tag{25}$$

then a solution of equation (24) is

$$F = \frac{1}{r} \int_{r'}^{r} G(\xi, t) d\xi. \tag{26}$$

A fourth solution of our equations is obtained by simply taking

$$\bar{H}^{(4)} = C\Delta \frac{1}{r}, \quad C = \text{Constant},$$

for

$$\nabla \cdot \left(\nabla \frac{1}{r}\right) = \nabla^2 \frac{1}{r} \equiv 0,$$

and

$$\nabla^2 \left(\nabla \frac{1}{r}\right) = \nabla \left(\nabla^2 \frac{1}{r}\right) \equiv 0.$$

With the aid of these four solutions, or linear combinations of them, one can solve some particular problems under suitable boundary conditions.

[1] E. N. Parker, "Interaction of the Solar Wind with the Geomagnetic Field," *Phys. Fluids*, **1**, 171–187, 1958.

[2] L. Brillouin, *L'Atome de Bohr* (Paris: Presses Universitaires de France, 1931), see pp. 125–127.

[3] J. L. Stratton, *Electromagnetic Theory* (New York: McGraw-Hill Book Co., 1941), pp. 183–184.

[4] *Ibid.*, pp. 131–135.

[5] S. I. Pai, "Energy Equation of Magneto-Gas Dynamics," *Phys. Rev.*, **105**, 1424–1426, 1957.

[6] C. W. Oseen, "Sur les formules de Green généralisées qui se présentent dans l'hydrodynamique et sur quelques-unes de leurs applications," *Acta Mathematica*, **34**, 205–284, 1910.

[7] Henri Villat, *Leçons sur l'hydrodynamique* (Paris: Gauthier-Villars, 1929), pp. 152–154.

DIRICHLET SPACES

BY A. BEURLING AND J. DENY

INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY

Communicated by Lars V. Ahlfors, August 7, 1958

This is an expository paper giving the definitions and the main properties of Dirichlet spaces. A first paper, devoted to the "elementary case," is to appear soon.[1] We shall refer to it as (AM). Complete proofs will be given later.

1. *Definitions.*—A *normalized contraction* T of the complex plane C is a transfor-

mation of C into itself which does not increase distances and keeps the origin fixed:

$$|T\xi' - T\xi''| \leq |\xi' - \xi''| \qquad (\xi', \xi'' \in C)$$
$$T(0) = 0.$$

Given a positive Radon measure, ξ, which is defined and everywhere dense on a locally compact Hausdorff space X ($\xi(\omega) > 0$ for each not empty open set $\omega \subset X$), we shall call *ξ-Dirichlet space* any Hilbert space $D = D(X, \xi)$ of complex valued functions $u(x)$ which are locally summable for ξ, the three following axioms being satisfied:

a) For each compact set $K \subset X$, there exists a number $A(K) > 0$ such that

$$\int_K |u| d\xi \leq A(K)||u||;$$

b) $\mathcal{C} \cap D$ is dense both in D and in $\mathcal{C}$;

c) For each normalized contraction T of C and each $u \in D$, the following holds:

$$Tu \in D \qquad \text{and} \qquad ||Tu|| \leq ||u||.$$

More precisely, elements in D are functions defined up to a set which is locally of zero ξ-measure (such a set is of zero ξ-measure if X is σ-compact). The norm in D is denoted by $||u||$, the associated scalar product by (u, v). $\mathcal{C}$ is the set of continuous complex valued functions on X, each of them having compact support. Axiom *b* means that there are many continuous functions in D. The following statement is equivalent to axiom *c:*

c') If u is in D, and if v is a function such that

$$|v(x) - v(y)| \leq |u(x) - u(y)|$$
$$|v(x)| \leq |u(x)|$$

for all $x, y \in X$, then v is in D, and $||v|| \leq ||u||$.

The classical "Dirichlet space" is generated by differentiable functions with compact support, in some Greenian domain of R^n, the norm being the square root of the Dirichlet integral $\int |\text{grad } u|^2 dx$.

2. *Potential Theory in Dirichlet Spaces.*—An obvious consequence of axiom *a* is the following:

LEMMA 1. *For each measurable bounded function f with compact support, there exists a unique element u_f in D such that*

$$(u_f, u) = \int f\bar{u}d\xi \qquad \text{for each } u \in D.$$

Such an element may be considered as the "potential generated by f"; more generally, a potential is an element u in D such that there exists a Radon measure μ on X satisfying

$$(u, \varphi) = \int \bar{\varphi} d\mu \qquad \text{for each} \qquad \varphi \in C \cap D. \tag{1}$$

If such a μ exists, it is unique. If this associated measure is positive, u is called a *pure potential.* Linear combinations of pure potentials are dense in D.

LEMMA 2. *Pure potentials are ≥ 0. An element $u \in D$ is a pure potential if and only if*

$$||u + v|| \geq ||u|| \qquad \text{for each} \qquad v \in D \qquad \text{with} \qquad R_e v \geq 0. \tag{2}$$

That an element u in D satisfying (2) is ≥ 0 follows from the convexity of the closed set

$$U = \{w: |R_e(w - u) \geq 0\};$$

in fact, u is the unique element of U having the smallest norm; but $|u|$ is in U, with norm $\leq ||u||$, according to axiom c or c'; hence $u = |u| \geq 0$. Now for a pure potential u, (2) is a consequence of (1) if $v \in \mathcal{C} \cap D$ (with $R_e v \geq 0$), and then, by an easy limit process, if $v \in D$ with $R_e v \geq 0$. Conversely, if $u \in D$ satisfies (2), (u, v) is a positive linear form on $\mathcal{C} \cap D$, which may be extended to $\mathcal{C}$, according to axiom b; hence there exists a positive μ such that (1) holds, and u is a pure potential.

Those definitions lead to a *kernel-free* potential theory. In this way the three next theorems are characteristic. The proofs given in (AM) for elementary Dirichlet spaces can be adjusted to our general case.

THEOREM 1 (CONDENSORS THEOREM). *Let ω_0 and ω_1 be two open sets with disjoint closures, ω_1 being bounded: then there exists a real potential u, with associated measure μ, such that*

(i) $0 \leq u(x) \leq 1$ *a.e.*[2]

(ii) $u(x) = 0$ *a.e. on* ω_0, $u(x) = 1$ *a.e. on* ω_1.

(iii) μ^+ *lies on* ω_1, μ^- *on* ω_0.

Taking for ω_0 the empty set, we get the *equilibrium theorem;* the equilibrium potential of a bounded open set ω is the unique element $u \in D$ having the smallest norm among those for which $R_e u(x) \geq 1$ a.e. on ω; it is a pure potential and the associated measure is called the equilibrium distribution of ω.

THEOREM 2. *The inferior envelope of two pure potentials is a pure potential.*

THEOREM 3 (BALAYAGE THEOREM). *Given a pure potential u_μ and an open set ω, there exists a pure potential $u_{\mu'}$ such that*

(i) *The associated measure μ' lies on $\bar{\omega}$ and* $\int d\mu' \leq \int d\mu$.

(ii) $u_{\mu'}(x) = u_\mu(x)$ *a.e. on* ω.

(iii) $u_{\mu'}(x) \leq u_\mu(x)$ *a.e.*

Let us now define the *capacity* of an open set ω as the infimum

$$\text{cap}(\omega) = \inf ||u||^2$$

for all the functions u in D with $R_e u \geq 1$ a.e. on $\omega(\text{cap}(\omega) = +\infty$ if there are no such functions. This may happen only if ω is unbounded.) Axiom c implies that, for each u in D and each number $\epsilon > 0$:

$$\text{cap}\,\{x: ||u(x)|| > \epsilon\} \leq ||u||^2/\epsilon^2. \qquad (3)$$

The exterior capacity of an arbitrary set e is defined as the infimum cap(e) of the capacities of open sets containing e.

Formula (3) permits us to refine functions of D. Let us state without proof:

THEOREM 4. *To each element $u \in D$ it is possible to associate a function u^* (refinement of u) such that:*

(i) $u^*(x) = u(x)$ *a.e. and vanishes outside some σ-compact set;*

(ii) *There exists a shrinking sequence of open sets with capacity tending to zero, u^* being continuous on the complement of each of them;*

(iii) *u^* is measurable with respect to the measure associated to any pure potential u_μ, and*

$$(u^*, u_\mu) = \int u^* d\mu.$$

Two refinements of the same u differ only on a set of zero exterior capacity. This notion of refined functions leads to a more precise potential theory, exceptional sets with respect to the ξ-measure being replaced by capacitary null sets.

3. *Representation of Dirichlet Spaces.*—One of the main problems of the whole theory is to find an explicit expression on the norm in a given Dirichlet space.

LEMMA 3. *Let f be given in $L^2 = L^2(\xi)$ or in D. For each number $\lambda > 0$ there exists a unique element $R_\lambda f$ in D which minimizes the quadratic functional*

$$F(u) = \lambda\|u\|^2 + \int |u - f|^2 d\xi; \tag{4}$$

$R_\lambda f$ is also the only element $u \in D$ such that u-f is in L^2 and

$$\lambda(u, v) + \int (u - f)\bar{v} d\xi = 0 \text{ for each } v \in L^2 \cap D. \tag{5}$$

The first part of Lemma 3 is a standard application of the identity

$$\frac{1}{2} F(u) + \frac{1}{2} F(v) - F\left(\frac{u + v}{2}\right) = \lambda \left\|\frac{u - v}{2}\right\|^2 + \int \left|\frac{u - v}{2}\right|^2 d\xi,$$

and (5) means that the variation of F vanishes for $u = R_\lambda f$.

LEMMA 4. *The operator R_λ: $f \to R_\lambda f$, defined in D and in L^2, has the following properties:*

(i) *R_λ is linear, positive hermitian, and bounded both in D and in L^2, with norm $\|R_\lambda\| \leq 1$; moreover, $f \in D$ and $\|R_\lambda f\| = \|f\|$ imply $f = 0$.*

(ii) *$\lim_{\lambda \to 0} R_\lambda = I$ (identity operator), $\lim_{\lambda \to \infty} R_\lambda = 0$, strongly in D as well as in L^2.*

(iii) *If the normalized contraction T of the complex plane leaves f invariant, the same holds for $R_\lambda f$.*

Let us sketch the proof. From characteristic property (5) we see at once that R_λ is linear and positive hermitian. Let f be given in D; writing $F(f) \geq F(R_\lambda f)$, we get

$$\lambda \|f\|^2 \geq \lambda \|R_\lambda f\|^2 + \int |R_\lambda f - f|^2 d\xi;$$

hence $\|R_\lambda f\| \leq \|f\|$ (equality holding if and only if $R_\lambda f = f$, which, according to (5), implies $f = 0$) and therefore $\|R_\lambda\|_D \leq 1$. Let f be given in L^2; taking $v = R_\lambda f$ in (5) and using Schwarz inequality, we get $\int |R_\lambda f|^2 \leq \int |f|^2 d\xi$. Thus (i) is proved.

The proof of (ii) is another easy consequence of Lemma 4. Finally, let f and T be as in the statement of (iii), and let us put $u = R_\lambda f$; then $f = Tf$ and axiom c imply

$$F(Tu) = \lambda \|Tu\|^2 + \int |Tu - Tf|^2 d\xi \leq \lambda\|u\|^2 + \int |u - f|^2 d\xi = F(u)$$

and therefore $Tu = u$, by the minimizing property of u.

LEMMA 5. *For each $\lambda > 0$, there exists a positive Radon measure α_λ on the product space $X \times X$ such that*

(i) *α_λ is symmetric;*

(ii) *The projection of α_λ on X is $\leq \xi(\alpha_\lambda(A \times X) \leq \xi(A)$ for each compact set $A \subset X$);*

(iii) *For each pair of functions $f, g \in L^2$,*

$$\int R_\lambda f \bar{g} d\xi = \iint f(x)\, \overline{g(y)} d\alpha_\lambda(x, y).$$

In fact, if $f \in \mathcal{C}$ is a real valued function such that $0 \leq f(x) \leq 1$, then $0 \leq R_\lambda f(x) \leq 1$ (a.e.) by Lemma 4 (iii), since $f = Tf$, T being the contraction of the complex plane which is the projection on the unit segment (0, 1) of the real axis. Thus the number $\sum \int R_\lambda f_i \bar{g}_i d\xi$ defines a *positive* linear form on the functions $\Sigma f_i \bar{g}_i$, which are dense in $\mathcal{C}(X \times X)$, and therefore a positive Radon measure α_λ on $X \times X$, satisfying (ii) and (iii). Part (i) follows from the hermitian property *of* R_λ.

THEOREM 5. *For each function $f \in D$ the norm is given by*

$$\|f\|^2 = \lim_{\lambda \to 0} (R_\lambda f, f) = \lim \|f\|_\lambda^2,$$

where

$$\|f\|_\lambda^2 = \frac{1}{\lambda}\left\{\int [1 - m_\lambda(x)]\,|f(x)|^2 d\xi + \frac{1}{2}\iint |f(x) - f(y)|^2\, d\alpha_\lambda(x, y)\right\}$$

and m_λ is the density of the projection of α_λ on X $(0 \leq m_\lambda \leq 1)$.

If $f \in \mathcal{C} \cap D$, this representation theorem is a consequence of formula (5), Lemma 4 (ii), and Lemma 5; extension to every $f \in D$ can be obtained by standard limit processes.[3]

Conversely, if a given function f is such that $\|f\|_\lambda$ is finite and bounded, we do not necessarily have $f \in D$; but this is true (and $\|f\| = \lim_{\lambda \to 0} \|f\|_\lambda$) under any one of the following additional assumptions:

$f \in L^2(\xi)$,

$f(x) \to 0$ when $x \to \infty$,

$|f|$ is $\leq$ some function of D.

If λ tends to zero, the measure $\mu_\lambda = \lambda^{-1}(1 - m_\lambda)\xi$ converges weakly to a measure μ, which is the limit of the equilibrium distribution of a bounded open set tending to the whole space X. At the same time $\sigma_\lambda = \alpha_\lambda/2\lambda$ converges outside the diagonal δ of $X \times X$ to a measure σ which can be defined directly (in $X \times X - \delta$) by

$$(f, g) = -2\iint f(x)\,\overline{g(y)}\, d\sigma(x, y),$$

where f and $g \in \mathcal{C} \cap D$ have supports without common points (if two such functions f and g are ≥ 0, axiom c implies $\|f - g\| \geq \|f + g\|$; hence $(f, g) \leq 0$, and there exists a positive σ in $X \times X - \delta$ with the prescribed property).

Those measures μ and σ lead to a more elegant expression of the norm of a function $f \in C \cap D$:

$$\|f\|^2 = \int |f(x)|^2 d\mu(x) + \iint |f(x) - f(y)|^2 d\sigma(x, y) + N(f),$$

where

$$N(f) = \lim_{\lambda \to 0} \iint |f(x) - f(y)|^2 d\sigma_\lambda(x, y) - \iint |f(x) - f(y)|^2 d\sigma(x, y) \geq 0$$

has the following *local* character: if $g \in \mathcal{C} \cap D$ is constant in some neighborhood of the support of $f \in \mathcal{C} \cap D$, then the associated bilinear form $N(f, g)$ vanishes.

A more explicit expression for the local part $N(f)$ of the square norm cannot be found unless additional assumptions are made on the structure of X. For instance, if X is a domain in R^m, if ξ is the Lebesgue measure on X, and if D contains all differentiable functions with compact support, there exist Radon measures ν_{ij} on X such that

$$N(f) = \sum_{i=1}^{m}\sum_{j=1}^{m}\int \frac{\partial f}{\partial x_i}\frac{\partial \bar{f}}{\partial x_j}\,d\nu_{ij}.$$

Second-order elliptic differential operators might be studied in this framework.

4. *Laplace Operator. Kernel.*—If an element $u \in D$ is the potential generated by the function f ($u = u_f$), we shall write $\Delta u = f$ and call f the Laplacian of u;[4] the domain of Δ is dense in D (Lemma 1).

For f given in D, $R_\lambda f$ is a solution of $u + \lambda\Delta u = f$ (after [5]). This fact, together with the formula

$$\lambda R_\lambda - \mu R_\mu = (\lambda - \mu)R_\lambda R_\mu \qquad (\lambda > 0, \mu > 0), \tag{6}$$

shows (Hille-Yoshida theorem) that a suitable restriction of $(-\Delta)$ is the infinitesimal generator of a semigroup of positive operators

$$T_t = \lim_{n\to\infty} (R_{t/n})^n \qquad (t > 0)$$

having the properties of R_λ which are listed in Lemma 4.

Another application of (6) is the following:

LEMMA 6. *Given a bounded measurable function f with compact support,*

$$u_f = \mathit{lim}_{\lambda\to\infty}\, \lambda R_\lambda f \qquad (\textit{strongly in } D).$$

The measure $\lambda\alpha_\lambda$ on the product space $X \times X$ (see Lemma 5) is increasing with λ and converges weakly to some positive symmetric measure κ on $X \times X$, called the *kernel* of D; the potential u_f can be expressed by means of the kernel as follows:

THEOREM 6. *Given a bounded measurable function f with compact support, the potential u_f is the density of the projection on X of the measure $f(y)\kappa(x, y)$.*

In other words:

$$\int u_f\bar{g}d\xi = \iint f(x)\,\overline{g(y)}\,d\kappa(x, y)$$

for each bounded measurable g with compact support; particularly,

$$\|u_f\|^2 = \iint f(x)\,\overline{f(y)}\,d\kappa(x, y).$$

This last formula proves that the kernel is a positive definite measure; the square norm of u_f can be interpreted as the "energy" of f, as well known in the classical Dirichlet space.[5]

5. *Dirichlet Rings.*—A corollary of the converse of Theorem 5 is the following useful generalization of axiom c':

LEMMA 7. *If $u_1, u_2, \ldots, u_n$ are elements in the Dirichlet space D, and if the function v satisfies*

$$|v(x)| \leq \Sigma\, |u_i(x)| \quad \textit{for each } x \in X,$$
$$|v(x) - v(y)| \leq \Sigma\, |u_i(x) - u_i(y)| \quad \textit{for each pair } x, y \in X,$$

then v is in D and $\|v\| \leq \Sigma\, \|u_i\|$.

It follows that the product of two bounded functions $u, v \in D$ is itself in D, and

$$\|uv\| \leq a\|v\| + b\|u\|, \tag{7}$$

where $a = \sup\, |u(x)|$, $b = \sup\, |v(x)|$.

If for each point $x \in X$ the capacity of the set $\{x\}$ is $\geq c^2 > 0$, each function of D is continuous (more precisely is a.e. equal to a continuous function: vanishing at infinity, and formula (3) implies $|u(x)| \leq ||u||/c$; therefore, according to (7),

$$||uv|| \leq \frac{2}{c} ||u|| \, ||v||.$$

Thus D is a normed ring (Dirichlet ring). There exist such rings, for instance the classical Dirichlet space on a finite interval of the real line. In such a ring the theorem of ideals holds:

THEOREM 7. *In a Dirichlet ring D on a separable space X, each closed ideal is a prime ideal, and is the set of functions of D vanishing on some closed set (therefore, it is the intersection of maximal ideals containing it).*

6. *Spectral Synthesis in Dirichlet Spaces.*—Given an element u in a Dirichlet space D, there exists a greatest open set, ω, having the following property:

$$(u, f) = 0 \text{ for each } f \in \mathcal{C} \cap D \text{ with support in } \omega.$$

Such a set is called a "set of regularity" of u; the complementary (closed) set is the *spectrum* of u, denoted $\sigma(u)$. The spectrum of a potential is obviously the support of the associated measure.

Let us state without any proof the following theorem of spectral synthesis:

THEOREM 8. *Each element $u \in D$ is the strong limit of linear combination of pure potentials having spectrum in $\sigma(u)$.*

It follows that the capacity of $\sigma(u)$ is zero if and only if $u = 0$.

Related to the last theorem is the "maximum-minimum principle," that we shall state only for a continuous function of D:

THEOREM 9. *If u is a continuous function of D, the image $u(X)$ is contained in the convex hull of the image of $\sigma(u)$ and of the origin of the complex plane.*

A corollary is the ordinary maximum principle: if $|u(x)| \leq a$ on $\sigma(u)$, then $|u(x)| \leq a$ everywhere.

7. *Special Dirichlet Spaces.*—Let us call negative definite function on a locally compact abelian group G any continuous complex valued function $\lambda(x)$ such that the hermitian form,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left[\lambda(x_i) + \overline{\lambda(x_j)} - \lambda(x_i - x_j) \right] \rho_i \bar{\rho}_j$$

is positive for each set of n points $x_1, x_2, \ldots, x_n \in G (n = 1, 2, \ldots)$.

A continuous function λ is negative definite if and only if $\lambda(0)$ is ≥ 0 and exp $(-t\lambda(x))$ is positive definite for each $t > 0$, i.e., if $\exp[-t(\lambda(x) - \lambda(0))]$ is the characteristic function of an infinitely divisible law of probability.

If $G = R^n$, this leads to the Levy-Khintchine representation formula, which we shall state for real (and then symmetric) λ:

$$\lambda(x) = C + Q(x) + \int (1 - \cos tx) \frac{1 + |t|^2}{|t|^2} d\sigma(t),$$

where C is a positive constant, Q a positive quadratic form, and σ a positive measure with finite total mass. This representation is unique under the assumption that σ is symmetric (with respect to the origin 0) and does not charge 0.

Now a Dirichlet space $D(X, \xi)$ is said to be *special* if X is a locally compact abelian group and ξ is the Haar measure on X, the following axiom being satisfied:

(d) If $U_s f$ is the function obtained from $f \in D$ by the translation $s \in X$ $(U_s f(x) = f(x - s))$, then $U_s f$ is in D, $\|U_s f\| = \|f\|$, and $U_s f$ is a continuous function of s.

Negative definite functions give explicit construction of all the special Dirichlet spaces:

THEOREM 10. *To each special Dirichlet space D on a locally compact abelian group X is associated a real valued negative definite function* λ *on the dual group* $\hat{x}$, *such that* $1/\lambda$ *is integrable on each compact neighborhood of the origin of* $\hat{x}$, *and*

$$|u|^2 = \int \lambda(\hat{x}) \; |\hat{u}(\hat{x})|^2 \, dx \text{ for all } u \in \mathcal{C} \cap D, \tag{8}$$

$\hat{u}$ *being the Fourier transform of u.*

Conversely such a negative definite function on X defines, by means of (8), a special Dirichlet space on X.

In this special case the kernel κ of D, defined on $X \times X$ (see 5) is invariant by the translations (s, s). The associated convolution kernel is the positive measure K(on X) having $1/\lambda$ as (generalized) Fourier transform. Each pure potential, with associated measure μ, is the density of the convolution $K^*\mu$.

If λ is a real valued negative definite function, λ^α is negative definite for each α, $0 \leq \alpha \leq 1$. Therefore, if K is the convolution kernel associated to a special Dirichlet space on X, there exists a family of such kernels K_α $(0 \leq \alpha \leq 1)$, with

$$K_1 = \text{K}, \qquad K_\alpha {}^* K_\beta = K_{\alpha+\beta}.$$

Taking $X = R^m$ $(m > 2)$ and $\lambda(\hat{x}) = |\hat{x}|^2$, K_α is the Riesz kernel of order 2α.

[1] A. Beurling and J. Deny, "Espaces de Dirichlet. I. Le cas élémentaire," *Acta Math.*, **99**, 1958.

[2] Except on a set which is locally of zero ξ-measure.

[3] In order to avoid some difficulties at the infinity, we may suppose, by Theorem 3, that each function of D vanishes out of some σ-compact set.

[4] In the classical Dirichlet space, Δ is the product of the ordinarily Laplace operator by some negative constant.

[5] According to the above definition, the kernel of the classical Dirichlet space (on a domain X) is the measure $G(x, y)\, \xi(x) \times \xi(y)$ on $X \times X$, G being the Green function of X.

A RELATIVIZED FATOU THEOREM

BY J. L. DOOB

UNIVERSITY OF ILLINOIS

Communicated December 11, 1958

1. *The Classical Fatou Boundary Limit Theory.*—Let R_N be an N-dimensional open solid sphere $(N \geq 2)$ of radius 1, with boundary R_N'. A function on R_N is said to have the nontangential limit b at a point η of R_N' if the function has the limit b in terms of approach in every cone of revolution with vertex η, axis the radius to η, semivertex angle less than $\pi/2$.

Let u be a function defined, harmonic, and positive on R_N. Then u corresponds