*DEGREE OF APPROXIMATION BY BOUNDED HARMONIC FUNCTIONS**

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The writer has recently published several notes on degree of approximation on a Jordan curve[1, 2] or arc[3] by polynomials in the complex variable, and also a paper on approximation by bounded harmonic functions,[4] all with emphasis on properties invariant under conformal transformation. The last-named paper admits of extensions (i) to more general degrees of approximation and (ii) to approximation on a Jordan arc; the object of the present note is briefly to set forth these extensions.

THEOREM 1. *Let* $\epsilon_1, \epsilon_2, \ldots$ *be a sequence of positive numbers approaching zero, where we suppose* $\epsilon_{[n/\lambda]} = O(\epsilon_n)$ *for every positive integral* λ *(here* $[m]$ *denotes the largest integer not greater than* m*) and where for every* r, $0 < r < 1$, *we have* $r^n = O(\epsilon_n)$. *Let* E *be an analytic Jordan curve in the z-plane containing the origin in its interior, and let* D *be a region containing* E. *Let the function* $u(z)$ *be defined on* E, *let the functions* $u_n(z)$ *be harmonic in* D, *and let* $(n = 1, 2, 3, \ldots)$

$$|u(z) - u_n(z)| \leqq A_1\epsilon_n, \qquad z \text{ on } E, \tag{1}$$

$$|u_n(z)| \leqq A_2R^n, \qquad z \text{ in } D, \tag{2}$$

be satisfied. Then there exist harmonic polynomials $p_n(z, 1/z)$ *in* z *and* $1/z$ *of respective degrees* n *such that*

$$|u(z) - p_n(z, 1/z)| \leqq A_3\epsilon_n, \; z \text{ on } E. \tag{3}$$

Here and below the letter A with subscript denotes a positive constant independent of n and z, a constant which may change from one usage to another.

Map one-to-one and conformally E onto the unit circumference E_1: $|w| = 1$ in the w-plane; then the map is one-to-one and conformal also in regions D_1 (a subregion of D) and D_2 containing E and E_1 respectively. As in the proof of Theorem 2.2 of reference 4, and by use of Lemma 1.2 of reference 4, there exist functions $\phi_n(w)$ *single-valued* and analytic in D_2 whose transforms $\Phi_n(z) \equiv \Phi_{n1}(z) + i\Phi_{n2}(z)$ satisfy the conditions $\Phi_{n1}(z) \equiv u_n(z)$ on E,

$$|\Phi_n(z)| \leqq A_4R_1{}^n, \qquad z \text{ in } D_3, \tag{4}$$

$$|u(z) - \Phi_{n1}(z)| \leqq A_1\epsilon_n, \qquad z \text{ on } E, \tag{5}$$

where D_3 is an annular subregion of D_1 containing E. In a suitably chosen annular subregion D_4 of D_3 containing E the components (defined in terms of the Cauchy integral) of the respective $\Phi_n(z)$ can be approximated by polynomials in z and polynomials in $1/z$, as in reference 2, in such a way that we have

$$|\Phi_n(z) - P_{nk}(z, 1/z)| \leqq A_5R_1{}^n\sigma_1{}^k, \qquad 0 < \sigma_1 < 1, \; z \text{ on } E, \tag{6}$$

where $P_{nk}(z, 1/z)$ is a polynomial in z and $1/z$ of degree k, and σ_1 is independent of k, n, and z. Here we choose the integer λ with $R_1\sigma_1{}^\lambda < 1$; it then follows from equations (5) and (6) that the real parts $p_{n,\lambda n}(z, 1/z)$ of the polynomials $P_{n,\lambda n}(z, 1/z)$ satisfy

$$|u(z) - p_{n,k}(z, 1/z)| \leqq A_6\epsilon_k, \qquad z \text{ on } E, \tag{7}$$

for the sequence $k = n, 2n, 3n, \ldots$. The $p_k(z, 1/z)$ may now be defined for every k by the equations

$$p_k(z, 1/z) \equiv p_{n,\lambda n}(z, 1/z), \qquad \lambda n \leqq k < (\lambda + 1)n,$$
$$p_k(z, 1/z) \equiv 0, \qquad 1 \leqq k < \lambda,$$

and equation (3) follows from (7) by the hypothesis on the ϵ_n.

A somewhat simpler result than Theorem 1 deserves to be stated:

COROLLARY. *Under the conditions of Theorem 1, let D contain both E and the interior of E. Then the polynomials $p_n(z, 1/z)$ may be chosen as harmonic polynomials in z so that equation (3) is valid.*

Under these conditions we map E and its interior onto E_1 and its interior; Jordan regions D_1, D_3, and D_4 may be chosen to contain E and its interior, and the $P_{nk}(z, 1/z)$ satisfying (6) may be chosen as polynomials in z alone, whose real parts $p_n(z, 1/z)$ are harmonic polynomials in z and satisfy equation (7). Then equation (3) follows as in the proof of Theorem 1.

The Corollary is in fact valid (compare reference 2) if D is an arbitrary Jordan region and if E is an arbitrary closed point set interior to D; we use Lemma 1.2 of reference 4, and conformal transformation onto the w-plane is not necessary in the proof.

The Corollary admits a converse:

THEOREM 2. *Let the ϵ_n satisfy the conditions of Theorem 1, let E be a Jordan curve, and let D be a bounded region containing E and its interior. If $u(z)$ is defined on E, and if harmonic polynomials $p_n(z)$ in z of respective degrees n exist satisfying*

$$|u(z) - p_n(z)| \leqq A_1\epsilon_n, \qquad z \text{ on } E, \tag{8}$$

then for suitably chosen R we have

$$|p_n(z)| \leqq A_2R^n, \qquad z \text{ in } D. \tag{9}$$

Thus the $p_n(z)$ satisfy the conditions prescribed for the $u_n(z)$ in the hypothesis of the Corollary.

It follows from (8) that the $p_n(z)$ are uniformly bounded on and interior to E. Let Γ be a circle interior to E, and let Γ_1 be a concentric circle whose radius is R times as large. If R is suitably chosen, D lies in Γ_1. The conclusion (9) now follows by Lemma 1.3 of reference 4, or by a more specific lemma due to Szegö (*loc. cit.*).

The hypothesis of Theorem 1 is invariant under one-to-one conformal (i.e. analytic) transformation of E, which implies also one-to-one conformal transformation of some region D containing E. The hypothesis remains invariant also if (as in the Corollary to Theorem 1 and in Theorem 2) the region D is required to contain both E and its interior. Under the latter condition the class of functions that can be approximated on E (not necessarily analytic) by harmonic polynomials in z of respective degrees n with degree of approximation $O(\epsilon_n)$ is also invariant under one-to-one conformal transformation of E and its interior; if E is an analytic Jordan curve, this class on E itself is thus the set of transforms of functions (necessarily continuous, so the Dirichlet problem for $|z| < 1$ has a solution) which can be approximated on $|z| = 1$ by trigonometric polynomials in $\theta = -i \log z$ of respective degrees n with degree of approximation $O(\epsilon_n)$.

A further remark is in order:

COROLLARY. *Under the conditions of Theoiem 1, the polynomials $p_n(z, 1/z)$ can be chosen as harmonic polynomials in z alone of respective degrees n.*

As in the proof of Theorem 1, map one-to-one and conformally the curve E and its interior onto E_1: $|w| = 1$ and its interior, say $w = \phi(z)$, $z = \psi(w)$. By the invariance of the hypothesis of Theorem 1 and by Theorem 1 itself, there exist harmonic polynomials $p_n(w, 1/w)$ in w and $1/w$ of respective degrees n such that

$$|u[\psi(w)] - p_n(w, 1/w)| \leqq A_3\epsilon_n, \qquad w \text{ on } E_1.$$

However, on E_1: $|w| \equiv \rho = 1$ a harmonic polynomial in w and $1/w$ of degree n is also a harmonic polynomial in w alone of degree n; for every k we have on E_1

$$\rho^k(a_{1k} \cos k\theta + b_{1k} \sin k\theta) + \rho^{-k}(a_{2k} \cos k\theta + b_{2k} \sin k\theta) \equiv \rho^k[(a_{1k} + a_{2k}) \cos k\theta + (b_{1k} + b_{2k}) \sin k\theta].$$

By the invariance of the property expressed by equation (8), the present corollary follows.

The proof just given shows also that *if E is the unit circumference, and if equation (3) is satisfied, then equation (8) is also satisfied if the $p_n(z)$ are suitably chosen.* The corresponding question is still open if E is an arbitrary analytic Jordan curve.

We turn now from approximation on Jordan curves and regions to approximation on Jordan arcs, as in reference 3.

THEOREM 3. *Let the ϵ_n satisfy the conditions of Theorem 1, let E be an analytic Jordan arc, and let D be a region containing E. Let the function u(z) be defined on E, let the functions $u_n(z)$ be harmonic in D, and let (1) and (2) be satisfied. Let $z = \phi(w)$ map E one-to-one and conformally onto E_0: $-1 \leqq w \leqq 1$. Then the function $u[\phi(\cos\theta)]$ can be approximated by trigonometric polynomials in θ of degree n with degree of approximation $O(\epsilon_n)$.*

Conversely, if u(z) is given on E and $u[\phi(\cos\theta)]$ can be approximated by trigonometric polynomials in θ of degree n with degree of approximation $O(\epsilon_n)$, then functions $u_n(z)$ harmonic in D exist satisfying (1) and (2); the $u_n(z)$ may be chosen as harmonic polynomials in z of respective degrees n.

Since the original conditions on the $u_n(z)$ are invariant under one-to-one conformal transformation of D, or of a subregion of D containing E, the method of proof of the Corollary to Theorem 1 shows that on E_0 the function $u[\phi(w)]$ can be approximated by harmonic polynomials in w of degree n with degree of approximation $O(\epsilon_n)$. On E_0 these harmonic polynomials of degree n are also polynomials of degree n in the real variable w, and the first part of Theorem 3 follows[3] by use of the classical transformation $w = \cos\theta$ as in the Bernstein-Jackson-Montel-de la Vallée-Poussin theory of trigonometric approximation.

To prove the converse, we note that $u[\phi(w)]$ can be approximated on E_0 by polynomials of degree n in the real variable w with degree of approximation $O(\epsilon_n)$, again by the classical theory of trigonometric approximation. On E_0 these polynomials are also polynomials in the complex variable w of degree n, and by the Bernstein lemma (§3 of reference 4) as proved by S. Bernstein these polynomials satisfy the analogue of equation (2) in an arbitrary bounded region D_1 containing

E_0. Thus the real parts of these polynomials in w are harmonic polynomials in w of respective degrees n which satisfy in the w-plane the analogues of equation (1) and equation (2). By the invariance under conformal transformation of the property expressed by equations (1) and (2) we deduce equations (1) and (2) in the main conclusion of the second part of Theorem 3, except that now D is the image in the z-plane of a suitably chosen region in the w-plane rather than an arbitrary region of the z-plane containing E. However, by the methods of proof of the Corollary to Theorem 1 and of Theorem 2 it follows that in equation (1) the $u_n(z)$ may be chosen as harmonic polynomials in z of respective degrees n, and equation (2) is valid in an arbitrary bounded region.

THEOREM 4. *Under the conditions and with the notation of Theorem 3 on the* ϵ_n, *D, and E, let s be arc-length measured along E algebraically from the mid-point of E, and let* $2l$ *be the total length of E. Then* $f(z)$ *can be approximated on E by functions* $u_n(z)$ *harmonic in D which satisfy (1) and (2) when and only when the function* $u(z) \equiv u_2(l \cos \theta_1) \equiv u_2(s)$ *can be approximated by trigonometric polynomials in* θ_1 *with degree of approximation* $O(\epsilon_n)$.

Theorem 4 follows from Theorem 3 of reference 3, which shows essentially that trigonometric approximation in θ of degree $O(\epsilon_n)$ implies trigonometric approximation in θ_1 of degree $O(\epsilon_n)$, and conversely.

It is proved in Theorem 2 that if the $u_n(z)$ are harmonic polynomials in z of respective degrees n, inequality (2) is a consequence of inequality (1), provided E is a Jordan curve. This conclusion is false if E is an analytic Jordan arc, as is shown by the counter-example E: $-1 \leqq z \leqq 1$, $u(z) \equiv 0$, $u_n(z) \equiv n^n xy$ for all n. The $u_n(z)$ are not uniquely determined by their values on E. Likewise if the $u_n(z)$ are harmonic polynomials in z and $1/z$ of respective degrees n and if E is E_1: $|z| = 1$, it is not true that (1) implies (2); a counter-example is $u_n(z) \equiv n^n (\rho^n \cos n\theta - \rho^{-n} \cos n\theta)$, where ρ and θ are the usual polar coordinates.

However, a harmonic polynomial $p_n(z, 1/z)$ in z and $1/z$ of degree n is not uniquely determined by its values on E_1. If a sequence $p_n(z, 1/z)$ of such (real) polynomials is bounded on E_1, each polynomial is a trigonometric polynomial on E_1, and can be expressed on E_1 as a polynomial $P_n(z, 1/z)$ in z and $1/z$ of degree n; the latter polynomials satisfy $|P_n(z, 1/z)| \leqq A_1R^n$, $R > 1$, in an arbitrary annulus $1/R \leqq |z| \leqq R$, and the real parts of the $P_n(z, 1/z)$ are harmonic polynomials in z and $1/z$ of degree n that satisfy this same inequality and coincide with $p_n(z, 1/z)$ on E_1.

Similarly, in the particular case that E is $-1 \leqq z \leqq 1$, a (real) harmonic polynomial $p_n(z)$ in z of degree n is on E also a polynomial in x (where $z = x + iy$) of degree n on E, thus a polynomial $P_n(z)$ in the complex variable z of degree n. When so interpreted, it follows from the original form of Bernstein's lemma as proved by S. Bernstein that $|P_n(z)| \leqq A_1$, z on E, implies $|P_n(z)| \leqq A_1R^n$, for z on and interior to the ellipse $|z^2 - 1| = (R + 1/R)/2$. The real part of $P_n(z)$ is a harmonic polynomial $q_n(z)$ of degree n which coincides with $p_n(z)$ on E and which satisfies $|q_n(z)| \leqq A_1R^n$ on and interior to that ellipse.

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[1] Walsh, J. L., *Math. Zeit.*, **72,** 47–52 (1959).

[2] Walsh, J. L., these PROCEEDINGS, **45,** 1528–1533 (1959).

[3] *Ibid.*, **46,** 981–983 (1960).

[4] Walsh, J. L., *Jour. de Math. pures et appl.*, **39,** (1960).