The quantity $a^2\lambda(a;\ \mu)$ is a monotonic increasing function of a and attains its minimum value at $a = 0$. Therefore, if we let

$$\Lambda(\mu) = \lim_{a\to 0} \frac{1}{a^2\lambda(a;\ \mu)} \tag{40}$$

the minimum field strength, H_{min}, required to stabilize the adverse flow,

$$\Omega = \Omega_1[1 - (1 - \mu)\zeta], \text{ and } \mu < 1, \tag{41}$$

is given by

$$\frac{H_{\text{min}}^2}{4\pi\rho} = \Lambda(\mu)[2\Omega_1^2(1 - \mu)R_0 d]. \tag{42}$$

10. A further fact may be noted. If Ω = constant, the characteristic values of σ ($= p + m\Omega$) can be determined readily in terms of the corresponding characteristic values,[5] σ_0, when $H = 0$. Thus

$$\sigma = \frac{1}{2}[\sigma_0 \pm \sqrt{(\sigma_0^2 + 4\Omega_A^2)}]. \tag{43}$$

* The research reported in this paper has in part been supported by the Geophysics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command, under Contract AF 19(604)-2046 with the University of Chicago.

[1] Chandrasekhar, S., "The Stability of Viscous Flow Between Rotating Cylinders in the Presence of a Magnetic Field," *Proc. Roy. Soc. (London)*, **A216** 293–309 (1953); also, Niblett, E. R., "The Stability of Couette Flow in an Axial Magnetic Field," *Canadian J. Phys.*, **36**, 1509–1525 (1958). These papers deal with the case of two cylinders rotated in the same direction; the case when they rotate in opposite directions has recently been investigated and will be published in due course.

[2] Velikhov, E. P., "Stability of an Ideally Conducting Liquid Flowing Between Cylinders Rotating in a Magnetic Field," *J. Exptl. Theoret. Phys.* (U.S.S.R.), **36**, 1398–1404 (1959).

[3] For the significance of this discriminant see Chandrasekhar, S., "The Hydrodynamic Stability of Inviscid Flow Between Coaxial Cylinders," these PROCEEDINGS, **46**, 137 (1960).

[4] I am grateful to Dr. H. W. Reid for communicating to me his unpublished results.

[5] These modes have been discussed by Lord Kelvin, "Vibrations of a Columnar Vortex," *Mathematical and Physical Papers, IV Hydrodynamics and General Dynamics* (Cambridge: The University Press, 1910) pp. 152–165; for a more recent account, see Bjerknes, V., J. Bjerknes, H. Solberg, and T. Bergeron, *Physikalische Hydrodynamik* (Berlin: Springer, 1933), chap. 11.

THE GEOMETRY OF QUANTUM STATES

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Communicated December 9, 1959

An earlier note[1] contains the initial stages in an evolution of the mathematical structure of quantum mechanics as the symbolic expression of the laws of microscopic measurement. The development is continued here. The entire discussion remains restricted to the realm of quantum statics which, in its lack of explicit reference to time, is concerned either with idealized systems such that all properties are unchanged in time or with measurements performed at a common time.

The uncontrollable disturbance attendant upon a measurement implies that the act of measurement is indivisible. That is to say, any attempt to trace the history of a system during a measurement process usually changes the nature of the measurement that is being performed. Hence to conceive of a given selective measurement $M(a', b')$ as a compound measurement is without physical implication. It is only of significance that the first stage selects systems in the state b', and that the last one produces them in the state a'; the interposed states are without meaning for the measurement as a whole. Indeed, we can even invent a nonphysical state to serve as the intermediary. We shall call this mental construct the null state 0, and write

$$M(a', b') = M(a', 0)M(0, b') \tag{1}$$

The measurement process that selects a system in the state b' and produces it in the null state,

$$M(0, b') = \Phi(b'),$$

can be described as the annihilation of a system in the state b'; and the production of a system in the state a' following its selection from the null state

$$M(a', 0) = \Psi(a'),$$

can be characterized as the creation of a system in the state a'. Thus the content of (1) is the indiscernibility of $M(a', b')$ from the compound process of the annihilation of a system in the state b' followed by the creation of a system in the state a',

$$M(a', b') = \Psi(a')\Phi(b'). \tag{2}$$

The extension of the measurement algebra to include the null state supplies the properties of the Ψ and Φ symbols. Thus

$$\Psi(a')^\dagger = \Phi(a'), \qquad \Phi(b')^\dagger = \Psi(b')$$

and

$$\Psi(a')\Psi(b') = \Phi(a')\Phi(b') = 0, M(a', b')\Phi(c') = \Psi(a')M(b', c') = 0, \tag{3}$$

whereas

$$M(a', b')\Psi(c') = <b' | c'> \Psi(a'), \Phi(a')M(b', c') = <a' | b'> \Phi(c'), \tag{4}$$

and

$$\Phi(a')\Psi(b') = <a' | b'> M(0).$$

The fundamental arbitrariness of measurement symbols expressed by the substitution

$$M(a', b') \rightarrow e^{-i\varphi(a')} M(a', b') e^{i\varphi(b')}, \tag{5}$$

implies the accompanying substitution

$$\Psi(a') \rightarrow e^{-i\varphi(a')} \Psi(a'), \qquad \Phi(b') \rightarrow e^{i\varphi(b')} \Phi(b'), \tag{6}$$

in which we have effectively removed $\varphi(0)$ by expressing all other phases relative to it.

The characteristics of the measurement operators $M(a', b')$ can now be derived from those of the Ψ and Φ symbols. Thus

$$M(a', b')^{\dagger} = \Phi(b')^{\dagger}\Psi(a')^{\dagger} = \Psi(b')\Phi(a') = M(b', a'),$$

and

$$tr\, M(b', a') = tr\, \Phi(a')\Psi(b') = < a' | b' >,$$

while

$$M(a', b')M(c', d') = M(a', b')\, \Psi(c')\Phi(d') = \\ <b' | c' > \Psi(a')\, \Phi\, (d') = <b' | c'> M(a', d').$$

In addition, the substitution (6) transforms the measurement operators in accordance with (5).

The various equivalent statements contained in (3) show that the only significant products—those not identically zero—are of the form $\Psi\Phi$, $\Phi\Psi$, and $X\Psi$, ΦX, in addition to XY, where the latin symbols are operators, elements of the physical measurement algebra. According to the measurement operator construction (2), all operators are linear combinations of products $\Psi\Phi$,

$$X = \sum_{a'b'} \Psi(a') < a' | X | b' > \Phi(b')$$

and the evaluation of the products $X\Psi$, ΦX, and XY reduces to the ones contained in (4),

$$\Psi(a')\Phi(b')\Psi(c') = \Psi(a') < b' | c' >,$$

$$\Phi(a')\, \Psi(b')\, \Phi(c') = < a' | b' > \Phi(c').$$

Hence, in any manipulation of operators leading to a product $\Phi\Psi$, the latter is effectively equal to a number,

$$\Phi(a')\Psi(b') = < a' | b' >,$$

and in particular

$$\Phi(a')\Psi(a'') = \delta(a', a''). \tag{7}$$

It should also be observed that, in any application of 1 as an operator we have, in effect

$$1 = \sum_{a'} M(a') = \sum_{a'} \Psi(a')\Phi(a').$$

Accordingly,

$$X = \sum_{a'b'} \Psi(a')\Phi(a')X\Psi(b')\Phi(b'),$$

which shows that

$$\Phi(a')\, X\Psi(b') = < a' | X | b' >.$$

The bracket symbols

$$<a' | = \Phi(a'), \qquad | b' > = \Psi(b')$$

are designed to make this result an automatic consequence of the notation (Dirac). In the bracket notation various theorems, such as the law of matrix multiplication, or the general formula for change of matrix representation, appear as simple applications of the expression for the unit operator

$$1 = \sum_{a'} |a' > < a'|.$$

We have associated a Ψ and a Φ symbol with each of the N physical states of a description. Now the symbols of one description are linearly related to those of another description,

$$\Psi(b') = \sum_{a'} \Psi(a')\Phi(a')\Psi(b') = \sum_{a'} \Psi(a') < a'|b' >, \tag{8}$$

and

$$\Phi(a') = \sum_{b'} < a'|b' > \Phi(b'), \tag{9}$$

which also implies the linear relation between measurement operators of various types. Arbitrary numerical multiples of Ψ or Φ symbols thus form the elements of two mutually adjoint algebras of dimensionality N, which are vector algebras since there is no significant multiplication of elements within each algebra. We are thereby presented with an N-dimensional geometry—the geometry of states—from which the measurement algebra can be derived, with its properties characterized in geometrical language. This geometry is metrical since the number $\Phi\Psi$ defines a scalar product. According to (7), the vectors $\Phi(a')$ and $\Psi(a')$ of the a-description provide an orthonormal vector basis or coordinate system, and thus the vector transformation equations (8) and (9) describe a change in coordinate system. The product of an operator with a vector expresses a mapping upon another vector in the same space,

$$X\Psi(b') = \sum_{a'}\Psi(a')\Phi(a')X\Psi(b') = \sum_{a'}\Psi(a') < a'|X|b' >,$$
$$\Phi(a')X = \sum_{b'} <a'|X|b'> \Phi(b').$$

The effect on the vectors of the a-coordinate system of the operator symbolizing property A,

$$A = \sum a'\Psi(a')\Phi(a')$$

is given by

$$A\Psi(a') = a'\Psi(a'), \ \Phi(a')A = \Phi(a')a',$$

which characterizes $\Psi(a')$ and $\Phi(a')$ as the right and left eigenvectors, respectively, of the complete set of commuting operators A, with the eigenvalues a'. Associated with each vector algebra there is a dual algebra in which all numbers are replaced by their complex conjugates.

The eigenvectors of a given description provide a basis for the representation of an arbitrary vector by N numbers. The abstract properties of vectors are realized by these sets of numbers, which are kownn as wave functions. We write

$$\Psi = \sum_{a'} |a' > <a'| \Psi$$
$$= \sum_{a} |a' > \psi(a').$$

and

$$\Phi = \sum_{a'} \phi(a') < a' |,$$
$$\phi(a') = \Phi |a' >.$$

If Φ and Ψ are in adjoint relation, $\Phi = \Psi^{\dagger}$, the corresponding wave functions are connected by

$$\phi(a') = \psi(a')^*.$$

The scalar product of two vectors is

$$\Phi_1 \Psi_2 = \sum_{a'} \Phi_1 |a' > <a' | \Psi_2 = \sum_{a'} \phi_1(a')\psi_2(a')$$

and, in particular,

$$\Psi^{\dagger}\Psi = \sum_{a'} \psi(a')^* \psi(a') \geq 0$$

which characterizes the geometry of states as a unitary geometry. The operator $\Psi_1\Phi_2$ is represented by the matrix

$$<a' | \Psi_1 \Phi_2 | b' > = \psi_1(a')\phi_2(b'),$$

and wave functions that represent $X\Psi$ and ΦX are

$$<a' | X\Psi = \sum_{b'} <a' | X | b' > \psi(b')$$

and

$$\Phi X | b' > = \sum_{a'} \phi(a') < a' | X | b' >.$$

On placing $X = 1$, we obtain the relation between the wave functions of a given vector in two different representations,

$$\psi(a') = \sum_{b'} <a' | b' > \psi(b')$$
$$\phi(b') = \sum_{a'} \phi(a') <a' | b' >.$$

From the viewpoint of the extended measurement algebra, ϕ and ψ wave functions are matrices with but a single row, or column, respectively.

It is convenient fiction to assert that every Hermitian operator symbolizes a physical quantity, and that every unit vector symbolizes a state. Then the expectation value of property X in the state Ψ is given by

$$<X>_{\Psi} = \Psi^{\dagger} X \Psi = \sum_{a'a''} \psi(a')^* <a' | X | a'' > \psi(a'').$$

In particular, the probability of observing the values a' in an A-measurement performed on systems in the state Ψ is

$$p(a', \Psi) = <M(a') >_{\Psi} = \Psi^{\dagger} |a'> < a' | \Psi = |\psi(a')|^2.$$

The geometry of states provides the elements of the measurement algebra with the geometrical interpretation of operators on a vector space. But operators considered in themselves also form a vector space, for the totality of operators is closed under addition and under multiplication by numbers. The dimensionality of this operator space is N^2 according to the number of linearly independent measurement symbols of any given type. A unitary scalar product is defined in the operator space by the number

$$<X|Y> = tr(X^{\dagger}Y) = <Y^{\dagger}|X^{\dagger}>,$$

which has the properties

$$\begin{aligned} <X|Y>^* &= <Y|X> \\ <X|X> &\geq 0. \end{aligned}$$

The trace evaluation

$$\text{tr } M(b', a')M(a'', b'') = \delta(a', a'')\delta(b', b'')$$

characterizes the $M(a', b')$ basis as orthonormal.

$$<M(a', b')|M(a'', b'')> = \delta(a'b', a''b''),$$

and the general linear relation between measurement symbols,

$$M(c', d') = \sum_{a'b'} <a'|c'><d'|b'> M(a', b'),$$

can now be viewed as the transformation connecting two orthonormal bases. This change of basis is described by the transformation function

$$<a'b'|c'd'> \equiv <M(a'b')|M(c', d')> = <a'|c'> <d'|b'>,$$

which is such that

$$<a'b'|c'd'> = <d'c'|b'a'>,$$

and

$$\begin{aligned} <a'b'|c'd'>^* &= <c'd'|a'b'> \\ &= <b'a'|d'c'>. \end{aligned}$$

One can also verify the composition property of transformation functions,

$$\sum_{c'd'} <a'b'|c'd'> <c'd'|e'f'> = <a'b'|e'f'>.$$

The probability relating two states appears as a particular type of operator space transformation function,

$$\begin{aligned} p(a', b') &= <a'|b'> <b'|a'> \\ &= <a'a'|b'b'>. \end{aligned}$$

Let $X(\alpha)$, $\alpha = 1 \ldots N^2$, be the elements of an arbitrary orthonormal basis,

$$<X(\alpha)|X(\alpha')> = \delta(\alpha, \alpha').$$

The connection with the $M(a', b')$ basis is described by the transformation function

$$\begin{aligned} <a'b'|\alpha> &= \text{tr } M(b', a')X(\alpha) \\ &= <a'|X(\alpha)|b'>. \end{aligned}$$

We also have

$$<\alpha|a'b'> = <a'b'|\alpha>^* = <b'|X(\alpha)^\dagger|a'>$$

and the transformation function property

$$\sum_\alpha <a'b'|\alpha> <\alpha|a''b''> = \delta(a'b', a''b'')$$

acquires the matrix form

$$\sum_\alpha <a'|X(\alpha)|b'> <b''|X(\alpha)^\dagger|a''> = \delta(a', a'')\delta(b', b'').$$

If we multiply the latter by the b-matrix of an arbitrary operator Y, the summation with respect to b' and b'' yields the a-matrix representation of the operator equation

$$\sum_\alpha X(\alpha)YX(\alpha)^\dagger = 1 \text{ tr } Y,$$

the validity of which for arbitrary Y is equivalent to the completeness of the operator basis $X(\alpha)$. Since the operator set $X(\alpha)^\dagger$ also forms an orthonormal basis we must have

$$\sum_\alpha X(\alpha)^\dagger YX(\alpha) = 1 \text{ tr } Y,$$

and the particular choice $Y = 1/N$ gives

$$\frac{1}{N}\sum_\alpha X(\alpha)X(\alpha)^\dagger = \frac{1}{N}\sum_\alpha X(\alpha)^\dagger X(\alpha) = 1.$$

The expression of an arbitrary operator relative to the orthonormal basis $X(\alpha)$,

$$X = \sum_\alpha X(\alpha)x(\alpha),$$

defines the components

$$x(\alpha) = <X(\alpha)|X> \equiv <\alpha|X>$$

For the basis $M(a', b')$, the components are

$$x(a'b') = tr\ M(b', a')X = <a'|X|b'>,$$

the elements of the ab-matrix representation of X. The scalar product in operator space is evaluated as

$$<X|Y> = \sum_\alpha x(\alpha)^*y(\alpha)$$

and

$$<X|X> = \sum_\alpha |x(\alpha)|^2 \geq 0.$$

On altering the basis the components of a given operator change in accordance with

$$x(\alpha) = \sum_\beta <\alpha|\beta> x(\beta).$$

For measurement symbol bases this becomes the law of matrix transformation.

There are two aspects of the operator space that have no counterpart in the state spaces—the adjoint operation and the multiplication of elements are defined in the same space. Thus

$$X(\alpha) = \sum_{\beta} (\alpha\beta) X(\beta)\dagger$$
$$(\alpha\beta) = (\beta\alpha) = \text{tr } X(\alpha)\ X(\beta),$$

and

$$X(\alpha)X(\beta) = \sum_{\gamma} (\alpha\beta\gamma) X(\gamma)\dagger$$
$$= \sum_{\gamma} X(\gamma) < \gamma | \alpha\beta >$$

where

$$(\alpha\beta\gamma) = (\beta\gamma\alpha) = (\gamma\alpha\beta)$$
$$= \text{tr } X(\alpha)X(\beta)X(\gamma)$$

and

$$<\gamma | \alpha\beta> = \text{tr } X(\gamma)\dagger X(\alpha)X(\beta).$$

Some consequences are

$$<\alpha | X>^* = \sum_{\beta} (\alpha\beta) < \beta | X\dagger>,$$
$$<\gamma | XY> = \sum_{\alpha\beta} <\gamma | \alpha\beta> <\alpha | X> <\beta | Y>,$$

which generalize the adjoint and multiplication properties of matrices. The elements of the operator space appear in the dual role of operator and operand on defining matrices by

$$<\alpha | X | \alpha''> = <\alpha | XX(\alpha'')>$$
$$= \sum_{\alpha'} <\alpha | \alpha'\alpha''> <\alpha' | X>.$$

The measurement symbol bases are distinguished in this context by the complete reducibility of such matrices, in the sense of

$$<a'b' | X | a''b'' > = <a' | X | a''> \delta(b', b'').$$

Otherwise expressed, the set of N measurement symbols $M(a', b')$, for fixed b', or fixed a', are left and right ideals, respectively, of the operator space.

The possibility of introducing Hermitian orthonormal operator bases is illustrated by the set

$$a' \neq a'': \quad \begin{matrix} 2^{-1/2}[M(a', a'') + M(a'', a')] \\ 2^{-1/2}\, i[M(a', a'') - M(a'', a')] \end{matrix}, \quad M(a').$$

For any such basis

$$<\alpha | \alpha'> = (\alpha\alpha') = \delta(\alpha, \alpha')$$

and

$$<\alpha | X>^* = <\alpha | X\dagger>,$$

which implies that a Hermitian operator X has real components relative to a Hermitian basis, and therefore

$$\langle X|X\rangle = \sum_{\alpha} x(\alpha)^2 \geq 0.$$

Thus the subspace of Hermitian operators is governed by Euclidean geometry, and a change of basis is a real orthogonal transformation,

$$X(\alpha) = \sum_{\beta} (\alpha\beta) X'(\beta).$$

When the unit operator (multiplied by $N^{-1/2}$) is chosen as a member of such bases it defines an invariant subspace, and the freedom of orthogonal transformation refers to the $N^2 - 1$ basis operators of zero trace.

Important examples of orthonormal operator bases are obtained through the study of unitary operators.

[1] Schwinger, J., these PROCEEDINGS, **45**, 1542 (1959).