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A MATHEMATICAL AID IN OPTIMIZING ENGINEERING DESIGNS

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Introduction and Results.—An engineer is frequently faced with the problem of optimizing a design to obtain a minimization of total operating costs. The writer has found that in an important class of such problems the desired minimum can be found directly without recourse to the laborious procedure of first solving for the optimum value of the parameters and then substituting back into the cost equation or to the soulless operation of a machine which gives numerical answers but no insight. The purpose of this note is to present this direct solution.

In the class of problems to which the present technique is applicable, the operation cost C is expressed as a polynomial of the independent parameter. The technique is restricted, however, to the case where the polynomial contains one more term than the number of independent parameters. Thus, if we denote our parameters by $x_1, x_2, \ldots x_\sigma$, we are to minimize the cost

$$C(x_1, x_2, \ldots x_\sigma) = \sum_{i=1}^{n} T_i$$

where

$$T_i = a_i \prod_{j=1}^{\sigma} x_j^{\beta_{ij}},$$

and where we have the restriction $n = \sigma + 1$.

The first step in our technique is to find a product of our n terms, each raised to an appropriate exponent α_i, which contains none of our σ parameters. This product we write as

$$\prod_{j=1}^{n} T_j^{\alpha_j} = K$$

Such a set of α's is unique, apart from a common factor. In order to render the α's completely unique, we impose a normalization condition

$$\sum_{=1}^{n} \alpha_i = 1.$$

Our technique is not applicable to those cases in which normalization cannot be imposed, as when the sum of α's is identically zero. Usually the appropriate set of α's can be found by mere inspection. Thus, in the example

$$C(x, y) = ax + b/xy + cy^2,$$

we have

$$(\alpha_1, \alpha_2, \alpha_3) = (2/5)(1, 1, 1/2),$$

and

$$K = a^{2/5}b^{2/5}c^{1/5}.$$

In more obtuse cases, we obtain the n α's by solving the n linear equations

$$\sum_{j=1}^{n} \beta_{ij}\alpha_j = 0, \quad i = 1, 2, \ldots \sigma$$

$$\sum_{j=1}^{n} \alpha_j = 1.$$

Our second and final step is to write our desired minimum cost as

$$C_{\min} = K/\prod_{i=1}^{n} \alpha_i^{\alpha_i}.$$

Analysis.—The crux to our solution lies in adopting the appropriate viewpoint. Since our cost C has been formulated in terms of the σ parameters $x_1, x_2, \ldots x_\sigma$, our instinct is to regard these x's as the independent variables with which we must work. But we may with equal justification regard the n T's as our independent variables, independent apart from the condition that the product K remain constant. Adopting this second viewpoint we may state our cost problem as the minimization of the function

$$C(T_1, T_2, \ldots T_n) = \sum_{j=1}^{n} T_j$$

subject to the condition that

$$K(T_1, T_2, \ldots T_n) = \prod_{j=1}^{n} T_j^{\alpha_j}$$

remain a constant K.

Our cost problem, formulated in terms of the T's. is directly soluble by the undetermined multiplier[1] method of Lagrange. Thus Lagrange has shown that a constant λ (the undetermined multiplier) exists such that that set of T's which minimizes $C(T_1, T_2, \ldots T_n)$ subject to the condition that $K(T_1, T_2, \ldots T_n)$ remain a constant, also minimizes

$$C - \lambda K$$

for completely arbitrary variations in the T's. By direct differentiation, we readily show that this last expression has a minimum for that set of T's given by

$$T_j = \alpha_j\lambda K,$$

and hence obtain

$$C_{\min} = \lambda K.$$

We may now determine the "undetermined multiplier" λ by substituting this solution for the T's into the equation which defines K. We obtain

$$\lambda = 1/\prod_{j=1}^{n} \alpha_i^{\alpha_j}.$$

Comments.—If our minimum cost is sufficiently low to justify economically the design concept leading to the function $C(x_1, x_2, \ldots x_\sigma)$, we of course wish to find that set of parameters $x_1, x_2, \ldots x_\sigma$ which leads to this minimum cost. From our prior solution for the T's,

$$T_i = \alpha_i \lambda K,$$

we may frequently, by mere inspection, write the solution for the optimum values of the x's. When inspection is not sufficient, we resort to a solution of a set of linear equations developed by A. Fein.[2] This set of equations, as well as other interesting deductions following from the above-described technique, are described by Fein in a forthcoming paper.

2. When our solution for $C_{\min}$ is complex, as is the case where one of the α's is negative and fractional, the cost has been improperly formulated in terms of the parameters $x_1, x_2, \ldots x_\sigma$. Thus, suppose we are presented with the cost function

$$C = ax^{-2} - bxy + cy^2.$$

Our analysis gives at once

$$C_{\min} = \sqrt{-1}\, a^{1/2} b c^{-1/2}.$$

We conclude our cost was incorrectly formulated.

3. We have restricted ourselves to the particular case

$$n = \sigma + 1.$$

We shall now demonstrate that no minimum occurs when

$$n < \sigma + 1.$$

Suppose, for example, that

$$n = \sigma.$$

We then proceed by regarding one of the σ parameters $x_1, x_2 \ldots x_\sigma$ as a constant, only $\sigma - 1$ as variables. We then employ the technique developed in this paper to find the minimum $C_{\min}$ with respect to these $\sigma - 1$ variables. This minimum will contain as a factor that x_i, raised to some power, which was regarded as a constant. Such a factor cannot be minimized with respect to x_i, apart from the trivial value of zero when x_i occurs as an even power.

4. The case $n > \sigma + 1$ remains unsolved.

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