*SOLUTION OF THE $\bar{\partial}$-NEUMANN PROBLEM ON STRONGLY PSEUDO-CONVEX MANIFOLDS**

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Introduction.—The main theorem of this note solves the $\bar{\partial}$-Neumann problem on strongly pseudo-convex manifolds. Only a brief outline of the proof is given here; details will be supplied elsewhere.[1] Using the main theorem we obtain a representation of the $\bar{\partial}$-cohomology by harmonic forms, thus generalizing the theory of harmonic integrals for compact manifolds. Another application which we obtain is a new proof of the theorem of Newlander and Nirenberg[2] on the existence of local holomorphic coordinates on an integrable almost-complex manifold.

A variant of the $\bar{\partial}$-Neumann problem was first formulated by Garabedian and Spencer.[3] The $\bar{\partial}$-Neumann problem in its present form was investigated by D. C. Spencer and the author[4] by means of integral equations. Morrey[5] solved the problem in the special case of (0,1)-forms on sufficiently small tubular neighborhoods of real analytic manifolds (obtained by complexification) by establishing certain *a priori* estimates; it is this approach which we adopt in our work.

Notation and Definitions.—Let M' be a complex analytic hermitian manifold of complex dimension n and let M be an open submanifold of M' such that bM, the boundary of M, is a closed C^∞ submanifold of M' and such that $\bar{M}$, the closure of M in M', is compact. Let $\mathfrak{A}$ be the space of C^∞ complex-valued forms on M'. Following the customary convention we say that a form ϕ is of type (p,q) if, in terms of local holomorphic coordinates, ϕ may be expressed in the form

$$\phi = \sum_{\substack{i_1<\ldots<i_p \\ j_1<\ldots<j_q}} \phi_{i_1\ldots i_p j_1\ldots j_q}\, dz^{i_1}\wedge\ldots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\ldots\wedge d\bar{z}^{j_q}.$$

We abbreviate our notation and write $\phi = \phi_{IJ}dz^I d\bar{z}^J$. We denote by $\mathfrak{A}^{p,q}$ the space of forms of type (p,q). Then the exterior derivative d maps $\mathfrak{A}^{p,q}$ into $\mathfrak{A}^{p+1,q} \oplus \mathfrak{A}^{p,q+1}$, and this induces the decomposition $d = \partial + \bar{\partial}$ where $\bar{\partial}:\mathfrak{A}^{p,q} \rightarrow \mathfrak{A}^{p,q+1}$. In terms of the hermitian metric we have the operator $*:\mathfrak{A}^{p,q} \rightarrow \mathfrak{A}^{n-q,n-p}$, the inner product

$$(\phi, \psi) = \int_M \phi\wedge *\bar{\psi}$$

and the norm $||\phi||^2 = (\phi, \phi)$. We denote by $\mathfrak{L}^{p,q}$ the Hilbert space of complex-valued norm-finite forms of type (p,q). The formal adjoint of $\bar{\partial}$, denoted by $\mathfrak{d}$, is defined by $\mathfrak{d} = -*\partial*$ and the Laplace operator $\square:\mathfrak{A}^{p,q}\rightarrow\mathfrak{A}^{p,q}$ is defined by $\square = \bar{\partial}\mathfrak{d} + \mathfrak{d}\bar{\partial}$. Let $r:M'\rightarrow R$ be the function such that $|r(P)|$ is the geodesic distance from P to bM and $r(P) > 0$ if $P\notin\bar{M}$, $r(P) < 0$ if $P\in M$. At each $P\in bM$ the space $\mathfrak{A}_P$ of forms evaluated at P decomposes into the space of normal forms, those divisible by dr, and its orthogonal complement, the tangential forms. If $\phi\in\mathfrak{A}$, $n\phi$ denotes the normal component of ϕ on bM. Let

$$\mathfrak{D}^{p,q} = \{\phi\in\mathfrak{A}^{p,q}|\ n\phi = 0 \text{ and } n\bar{\partial}\phi = 0 \text{ on } bM\}$$

and let $\square'$ be the closure of $\square$ restricted to $\mathfrak{D}^{p,q}$. We denote the domain of $\square'$ by $\mathfrak{D}^{p,q}$ and let $\mathfrak{H}^{p,q} = \{\phi\epsilon\ \mathfrak{D}^{p,q} \mid \square'(\phi) = 0\}$.

Finally we recall that M is called *strongly pseudo-convex* if whenever f is a local C^∞ real-valued function such that, $df \neq 0$, $f(P) < 0$ if $P\epsilon M$ and $f(P) > 0$ if $P \notin \bar{M}$ then if $0 \neq (a^1,\ldots,a^n)\epsilon\ \mathbf{C}^n$ and $\Sigma f_{z^i}a^i = 0$ we have $\Sigma f_{z^i\bar{z}^j}a^i\bar{a}^j > 0$.

Representation of $\bar{\partial}$-Cohomology.—As a consequence of the main theorem we can define (on a strongly pseudo-convex manifold M) the operator $N:\mathfrak{L}^{p,q} \to \mathfrak{D}^{p,q}$ by $N\square'\psi = \psi$ if $\psi\epsilon\mathfrak{D}^{p,q}$ and $\psi \perp \mathfrak{H}^{p,q}$, and by $N\phi = 0$ if $\phi\epsilon\mathfrak{H}^{p,q}$. The main theorem and the interior differentiability theorem for elliptic systems imply the following:

PROPOSITION. *The operator N defined above has the following properties:* (a) *N is bounded and, if $q \neq 0$, it is completely continuous.* (b) $\bar{\partial}N = N\bar{\partial}$. (c) *If $\phi\epsilon\mathfrak{L}^{p,q}$ and is of class C^∞, then $\phi = \bar{\partial}\mathfrak{b}N\phi + \mathfrak{b}\bar{\partial}N\phi + \gamma$ where $\gamma\epsilon\mathfrak{H}^{p,q}$ and each term is C^∞ on M.*

To study the $\bar{\partial}$-cohomology by means of forms in $\mathfrak{L}^{p,q}$ we first observe that, if M is strongly pseudo-convex, then there exists a manifold $\tilde{M}$ such that $M \subset \bar{M} \subset \tilde{M} \subset M'$ and such that the restriction map $H^q(\tilde{M}, \Omega^p) \to H^q(M, \Omega^p)$ is surjective for $q > 0$ (see Grauert[6]). Here Ω^p is the sheaf of germs of holomorphic $(p, 0)$-forms. Our main inequality also implies that if $\phi = \bar{\partial}\alpha$ and $\phi\epsilon\mathfrak{L}^{p,q}$ for $q\neq 0$, then there exists $\beta\epsilon\mathfrak{L}^{p,q-1}$ such that $\phi = \bar{\partial}\beta$. By the isomorphism theorem of Dolbeault, there is a form on M in each $\bar{\partial}$-cohomology class in M $(q \neq 0)$. Hence we obtain

PROPOSITION. *If $q > 0$, then $\mathfrak{H}^{p,q} \approx H^q(M, \Omega^p)$.*

Applications to Deformations of Complex Structure.—Let M_t', $t\epsilon(-1,1)$, be a differentiable family of complex structures on a differentiable manifold (see Kodaira and Spencer[7]). Suppose that $M_0 \subset M_0'$ is strongly pseudo-convex. Then for small t, $M_t \subset M_t'$ is strongly pseudo-convex, so the main theorem holds and the constants in the inequalities can be chosen independent of t.

PROPOSITION. *If f_0 is holomorphic on M_0' and if $\mathfrak{H}_0^{0,1} = 0$ then, for small t, there exists a function f_t which is holomorphic on M_t such that f_t and its first derivatives are continuous in t.*

The function f_t is defined by $f_t = f_0 - g_t$, where $g_t = \mathfrak{b}_t N_t \bar{\partial}_t f_0$. Our inequalities imply that $\lim_{t \to 0} ||g_t|| = 0$. We observe that $\square_t g_t = \square_t f_0$; thus the continuity in t follows from the L_p estimates given by Agmon, Douglis, and Nirenberg.[8]

It is easily shown that the above proposition holds for integrable almost-complex manifolds. If P is a point of an integrable almost-complex manifold, then for each $t\epsilon(0, 1)$ there exists a neighborhood V_t of P such that $\cap V_t = \{P\}$ and such that there exists a diffeomorphism λ_t: $V_t \to B$, where B is the unit ball in the holomorphic tangent space. Furthermore the λ_t and V_t can be so constructed that B_t, the almost-complex structure induced by λ_t, is part of a differentiable family of integrable almost-complex structures on B which includes the complex structure B_0. Applying the above proposition to B_t and taking for f_0 the coordinate functions on B_0, we obtain holomorphic coordinates on B_t for sufficiently small t. This argument gives a new proof of the result of Newlander and Nirenberg[2] along lines similar to those originally suggested by Spencer.[9] That is, we obtain the

PROPOSITION. *If M is an integrable almost-complex manifold, then each point of M has a neighborhood on which there exist holomorphic coordinates.*

The Principal Results.—The following proposition states the "weak decomposition" and it is proved by showing that $\square' = TT^* + T^*T$, where T denotes the closure of $\bar{\partial}$ in $\mathfrak{L}^{p,q}$. The self-adjointness then follows as in the work of Gaffney.[10]

PROPOSITION. *The operator* $\square'$: $\mathfrak{D}^{p,q} \rightarrow \mathfrak{L}^{p,q}$ *is self-adjoint and* $\mathfrak{L}^{p,q}$ *has the following orthogonal decomposition:* $\mathfrak{L}^{p,q} = [\square'\mathfrak{D}^{p,q}] \oplus \mathfrak{H}^{p,q}$. *Here we mean by* $[\mathcal{S}]$ *the closure of* $\mathcal{S}$.

The Kähler metric whose existence is asserted in the proposition below is constructed by taking a real-valued function f which defines the boundary globally and observing that (because of strong pseudo-convexity) the form $(\exp(Af))_{z^i\bar{z}^j}$ is positive definite for sufficiently large A.

PROPOSITION. *If* $M \subset M'$ *is strongly pseudo-convex, then there exists a Hermitian metric on* M' *which is Kähler in a neighborhood of* bM.

MAIN THEOREM. *If* $M \subset M'$ *is strongly pseudo-convex with a Hermitian metric on* M' *which is Kähler in a neighborhood of* bM, *then* $\mathfrak{L}^{p,q} = \square'\mathfrak{D}^{p,q} \oplus \mathfrak{H}^{p,q}$. *Furthermore,* $\mathfrak{H}^{p,q}$ *is finite dimensional, if* $q \neq 0$.

The proof of the theorem depends on the inequality (*) given in the following lemma. This inequality is also important for other applications.

LEMMA. *Under the hypothesis of the theorem and if* $q \neq 0$, *then there exists* $C > 0$ *such that for all* $\phi \epsilon \mathfrak{D}^{p,q}$ *we have*

$$(*) \qquad (\square\phi, \phi) + \|\phi\|^2 \geq C(\|\phi\|_{\bar{z}}^2 + \int_{bM} |\phi|^2 * dr + \|\phi\|^2),$$

where $\|\phi\|_{\bar{z}}^2$ *is defined by means of a fixed covering* $\{U_\alpha\}$ *by holomorphic coordinate neighborhoods,*

$$\|\phi\|_{\bar{z}}^2 = \sum_{\alpha,I,J,k} \int_{M \cap U_\alpha} |\phi_{IJ,\bar{z}_\alpha^k}|^2 *(1).$$

Outline of the Proof: To establish (*) we cover $\bar{M}$ with finitely many coordinate neighborhoods each of diameter less then ρ, where ρ is a fixed small number. Now, by use of a partition of unity, it suffices to prove (*) for forms $\phi \epsilon \dot{\mathfrak{D}}^{p,q}$ whose supports lie in a single coordinate neighborhood U_α. If U_α does not intersect bM then the inequality is obtained by standard techniques of elliptic equations (see, for example, Nirenberg[11]). If U_α intersects bM, we choose a holomorphic coordinate system on U_α whose origin is on bM and such that at the origin $g_{ij}(0) = 0$ and $(dg_{ij})_0 = 0$, where g_{ij} are the components of the metric tensor (such a coordinate system exists because of the Kähler property). We also choose the coordinates so that $r_{z^i\bar{z}^j}(0)$ is diagonal. Let $D(\phi)^2 = (\square\phi, \phi) + \|\phi\|^2$ and

$$E(\phi)^2 = \|\phi\|_{\bar{z}}^2 + \int_{bM} |\phi|^2 * dr + \|\phi\|^2.$$

Note that if $\phi \epsilon \dot{\mathfrak{D}}^{p,q}$, then

$$D(\phi)^2 = \|\bar{\partial}\phi\|^2 + \|\mathfrak{d}\phi\|^2 + \|\phi\|^2$$

and

$$\|\bar{\partial}\phi\|^2 = \sum_{IJk} \int_M |\phi_{IJ,\bar{z}^k}|^2 g^{(IJk)(IJk)} *(1) + \sum_{(IJk)\neq(PQm)} \int_M \phi_{IJ,\bar{z}^k} \bar{\phi}_{PQ,\bar{z}^m} g^{(IJk)(PQm)} *(1),$$

where the $g^{(IJk)(PQm)}$ are components of the metric tensor on $\mathfrak{A}^{p,q+1}$. Let $\phi^{IJ} = \phi_{KL} g^{(KL)(IJ)}$; then we obtain (for q>0)

$$\|\bar{\partial}\phi\|^2 + C_1\|\phi\|^2 \geq C_2\|\phi\|^2_{\bar{z}} - \sum_{kH} \epsilon^{kH}_{\langle kH\rangle}\, \epsilon^{mH}_{\langle mH\rangle} \int_M \phi^{I\langle mH\rangle}_{\bar{z}k}\, \bar{\phi}^{I\langle kH\rangle}_{zm} *(1),$$

where H runs over increasingly ordered $(q - 1)$-tuples, $\langle mH\rangle$ is the increasingly ordered q-tuple whose elements are $\{m\} \cup H$ and $\epsilon^{mH}_{\langle mH\rangle}$ is zero if $\langle mH\rangle$ has any repeated integers, otherwise it is the sign of the permutation that sends mH onto $\langle mH\rangle$. Integrating by parts, we obtain

$$\int_M \phi^{I\langle mH\rangle}_{\bar{z}k} \bar{\phi}^{I\langle kH\rangle}_{zm} *(1) = \int_{bM} \phi^{I\langle mH\rangle} [\bar{\phi}^{I\langle kH\rangle}_{zm} r_{\bar{z}k} + \bar{\phi}^{I\langle kH\rangle}_{\bar{z}k} r_{zm}] *dr - \int_M \phi^{I\langle mH\rangle}_{zm} \bar{\phi}^{I\langle kH\rangle}_{\bar{z}k} *(1) + 0(\rho)E(\phi)^2.$$

The boundary condition $n\phi = 0$ implies

$$\sum_k \epsilon^{kH}_{\langle kH\rangle} \bar{\phi}^{I\langle kH\rangle} r_{\bar{z}k} = 0 \quad \text{on } bM.$$

Differentiation with respect to z^m gives

$$\sum_k \epsilon^{kH}_{\langle kH\rangle} [\bar{\phi}^{I\langle kH\rangle}_{zm} r_{\bar{z}k} + \bar{\phi}^{I\langle kH\rangle} r_{\bar{z}kzm}] = \lambda r_{zm}$$

on bM. Multiplying by $\epsilon^{mH}_{\langle mH\rangle}\phi^{I\langle mH\rangle}$, summing on m and using the strong pseudo-convexity we obtain

$$- \sum \epsilon^{kH}_{\langle kH\rangle} \epsilon^{mH}_{<mH>} \phi^{I\langle mH\rangle} \bar{\phi}^{I<kH>}_{zm} \geq C_3|\phi|^2$$

on bM. Furthermore

$$\|\mathfrak{d}\phi\|^2 \geq C_4\Big|\sum_{kmH} \epsilon^{kH}_{\langle kH\rangle} \epsilon^{mH}_{<mH>} \phi^{I\langle mH\rangle}_{zm} \bar{\phi}^{I<kH>}_{\bar{z}k} *(1)\Big| + 0(\rho)E(\phi)^2.$$

Combining the above we obtain the inequality (*).

To complete the proof of the theorem we use the following inequality which is proved by Fourier transform methods analogous to those in Peetre's thesis.[12] For $a\epsilon\mathbf{R}$, let $bM_a = \{P\epsilon M | r(P) = a\}$. Then there exists $\epsilon > 0$ and $C > 0$ such that for any $\phi\epsilon\mathfrak{A}$,

$$\int_{bM_a} |\phi|^2 *dr \leq CE(\phi)^2,$$

where $|a| < \epsilon$. This inequality, together with the usual interior estimates, implies that E is completely continuous with respect to $\| \; \|$. Then the theorem for $q \neq 0$ follows by standard Hilbert space theory. The case $q = 0$ follows from the inequality $\|\Box\phi\| \geq K\|\phi\|$ for $\phi\epsilon\mathfrak{D}^{p,0}$ and ϕ orthogonal to $\mathfrak{H}^{p,0}$. This inequality is established by approximating ϕ with $\Box\alpha$, where $\alpha\epsilon\mathfrak{D}^{p,0}$. Then, using (*) on $\bar{\partial}\phi$ and $\bar{\partial}\alpha$, we obtain

$$\|\phi\|^2 \leq \|\bar{\partial}\phi\|\,\|\bar{\partial}\alpha\| + \|\phi\|\,\|\phi - \Box\alpha\| \leq C\|\Box\phi\|\,\|\Box\alpha\| + \|\phi\|\,\|\phi - \Box\alpha\|$$

which gives the result.

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EXAMPLES FOR DIFFERENTIABLE GROUP ACTIONS ON SPHERES

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1. *Introduction.*—The purpose of this note is to prove the following:

THEOREM. *Let G be a compact Lie group, containing more than one element. Then there exists a positive integer k such that G has an infinite number of differentiable actions on the k-sphere S^k (in its usual differentiable structure), no two of which are equivalent.*

Here differentiability is understood in the C^∞-sense; two differentiable actions are equivalent if there exists a diffeomorphism of S^k carrying one action into the other. G is not assumed connected, and may in particular be finite. Actually the actions to be constructed for the proof of the theorem are pairwise topologically inequivalent; that is, there is no homeomorphism of S^k carrying one action into the other. For the proof of the theorem, including the statement of the last sentence, we shall construct, for a given G and suitable k, a sequence of actions with the property that the stationary sets of any two actions are not homeomorphic, by virtue of having different fundamental groups (the stationary set of an action consists of all points left fixed by all elements of the group). Our construction is a modification of one of J. H. C. Whitehead's,[11] which in turn is based on an example of Newman's.[6] In the theorem above, the infinity cannot be more than countable by a result of Palais.[7]

2. *The manifolds Y_i and M_i.*—Our main observations in this section are that Newman's example can easily be modified to yield a sequence of different examples and that the manifolds in question can be taken differentiable.