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⁷ Ash, M. E., "The basic estimate of the $\bar{\partial}$ -Neumann problem in the non-kählerian case," (to appear).

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ON THEORIES CATEGORICAL IN UNCOUNTABLE POWERS

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Suppose Σ is a theory formalized in some countable, first-order (with equality) language, L . One says Σ is *categorical in power* K if Σ has a model of power K and all models of Σ of power K are isomorphic.¹ For example, the theory of dense linearly ordered sets without end points is categorical in power \aleph_0 but is not categorical in any higher power. On the other hand, the theory of algebraically closed fields of characteristic O is not categorical in power \aleph_0 but is categorical in every higher power. In 1954 Łoś raised the following question: *Is a theory categorical in one uncountable power necessarily categorical in every uncountable power?* We give an affirmative answer to this question.

Several recent papers² have pointed out the close connections between the properties of models of a theory and certain Boolean algebras of formulas. The isolated points in the dual spaces of these Boolean algebras play a particularly important role. We consider a certain class of such spaces and a certain category of continuous onto maps between them. The crux of our proof is an analog for such categories of the Cantor-Bendixson theorem. We also use methods related to those of Ehrenfeucht and Mostowski³ and the previously cited paper² of Vaught.

1. There will be no loss of generality in assuming Σ to be a complete theory in L . Suppose A is a model of Σ and $X \subseteq |A|$. (The set $|A|$ is the *universe* of A , i.e., A consists of the set $|A|$ and a set of relations on $|A|$.) We define an enlarged language, $L(A, X)$ by adding to L a new individual constant \bar{x}_A for each $x \in X$. ($(A, x)_{x \in X}$ is the model formed from A by taking each $x \in X$ as a distinguished element.) Denote by $\Sigma(A, X)$ the set of sentences of $L(A, X)$ which are valid in $(A, x)_{x \in X}$. Then $\Sigma(A, X)$ is obviously a complete theory in $L(A, X)$. We identify formulas of $L(A, X)$ which are equivalent in the theory $\Sigma(A, X)$ and denote by $F(A, X)$ the set of formulas having no free variable except some fixed one, say v_0 . We may consider $F(A, X)$ as a Boolean algebra with the Boolean operations of \cap , \cup , and $'$ corresponding to the logical connectives \wedge , \vee , and \sim , respectively.

By the celebrated result of Stone⁴ the set of maximal dual ideals of $F(A, X)$ form a totally disconnected compact space which we denote by $S(A, X)$. The space $S(A, X)$ has a basis of power $\leq \bar{X} + \aleph_0$. The points of $S(A, X)$ may be thought of as the kind of elements which may appear in models of $\Sigma(A, X)$. In particular, for each $a \in A$ there is a $p_a \in S(A, X)$ such that p_a consists of precisely those formulas of $F(A, X)$ satisfied by a in $(A, x)_{x \in X}$. We say a *realizes* p_a in A .

If A is a model of Σ , we say A is *saturated* if for every $X \subseteq |A|$ with $\overline{X} < \overline{A}$ every $p \in S(A, X)$ is realized in A . It can be shown⁵ that:

THEOREM 1. *If A and B are saturated models of Σ having the same power then A is isomorphic to B .*

The proof involves extending isomorphisms between subsystems one element at a time.

Vaught recognized that this theorem could be used in considering the problem of categoricity in power, and he showed⁵ (assuming the generalized continuum hypothesis) that if a theory is categorical in an increasing sequence of powers then it is categorical in the limit power.

Suppose A and B are models of Σ , $X \subseteq |A|$ and $Y \subseteq |B|$, and f maps X into Y . We say f is an *elementary monomorphism* of (A, X) into (B, Y) if whenever $n \in \omega$, ψ is a formula of L with free variables among v_0, \dots, v_{n-1} and $x_0, \dots, x_{n-1} \in X$, then $\vdash_A \psi(x_0, \dots, x_{n-1})$ if and only if $\vdash_B \psi(f(x_0), \dots, f(x_{n-1}))$. In particular, if $A \subseteq B$ and the identity map of A into B is an elementary monomorphism we say B is an *elementary extension* of A ($A < B$). One advantage of dealing only with elementary monomorphisms is that they have the *amalgamation property*; that is, if A and B_i ($i \in I$) are models of Σ , $X \subseteq |A|$, $Y_i \subseteq |B_i|$ ($i \in I$) and $f_i: (A, X) \rightarrow (B_i, Y_i)$ ($i \in I$) are elementary monomorphisms, then there is a model C of Σ , and a set $Z \subseteq |C|$ and elementary monomorphisms $g_i: (B_i, Y_i) \rightarrow (C, Z)$ ($i \in I$) such that for all $i, j \in I$, $g_i f_i = g_j f_j$.

It is easy to see that an elementary monomorphism $f: (A, X) \rightarrow (B, Y)$ induces a Boolean monomorphism of $F(A, X)$ into $F(B, Y)$ (by mapping \bar{x}_A to $f(\bar{x})_B$), which, by Stone's results induces a continuous onto map $f^*: S(B, Y) \rightarrow S(A, X)$. Thus with each complete theory Σ we may associate: (1) a class (denoted by $S(\Sigma)$) of topological spaces—namely, the class of all spaces $S(A, X)$ where A is a model of Σ and $X \subseteq |A|$; and (2) the category, $\mathcal{C}(\Sigma)$, of those continuous onto maps between members of $S(\Sigma)$ which are induced by elementary monomorphisms. It is to this class of spaces and category of maps that we wish to apply an analysis similar to that of a single space in terms of its derived sequence of spaces. (That a single Boolean algebra could be analyzed in terms of the derived sequence of its dual space was explicitly recognized by Mostowski and Tarski⁶). To do this we define a notion stronger than that of an isolated point: a point $p \in S \in S(\Sigma)$ is *algebraic* if for every $f \in \mathcal{C}(\Sigma)$ with range S , $f^{-1}(p)$ is a set of isolated points. (The term "algebraic" is suggested by the case where Σ is the theory of algebraically closed fields of characteristic 0. It is also consistent with a generalized notion of algebraic extension considered by Jónsson.)

We define $S' = S - \{\text{algebraic points of } S\}$. It turns out (essentially because all the spaces are compact and $\mathcal{C}(\Sigma)$ satisfies the dual of the amalgamation property) that if $(f: S_1 \rightarrow S_2) \in \mathcal{C}(\Sigma)$ then $f(S'_1) = S'_2$. Therefore, we may define $S'(\Sigma) = \{S'; S \in S(\Sigma)\}$ and $\mathcal{C}'(\Sigma) = \{(f: S'_1 \rightarrow S'_2); (f: S_1 \rightarrow S_2) \in \mathcal{C}(\Sigma)\}$. We may now repeat this process and find the algebraic points of S' (considered now as a member of $S'(\Sigma)$). If p is algebraic in S' we say it is *transcendental in rank 1* in $S(p \in Tr^1(S))$. We define S^2 as $S' - Tr^1(S)$ and $S^2(\Sigma) = \{S^2; S \in S(\Sigma)\}$. We proceed inductively to define:

$$S^\alpha = S - \bigcup_{\beta < \alpha} Tr^\beta(S)$$

$$Tr^\alpha(S) = \{ \text{algebraic points of } S^\alpha \text{ (considered as a member of } S^\alpha) \}$$

$$S^\alpha(\Sigma) = \{ S^\alpha; S \in \mathcal{S}(\Sigma) \}.$$

It may happen that for some ordinal α_0 and some $S \in \mathcal{S}(\Sigma)$, S^{α_0} is empty. It will then follow that for every $S \in \mathcal{S}(\Sigma)$, S^{α_0} is empty. (Again, this is essentially a consequence of the amalgamation property.) In this case we say that Σ is *totally transcendental* (because every $p \in S \in \mathcal{S}(\Sigma)$ is transcendental in some rank).

THEOREM 2. *If Σ is totally transcendental and $S(A, X) \in \mathcal{S}(\Sigma)$, then $\overline{S(A, X)} \leq \bar{X} + \aleph_0$.*

Proof: The space $S(A, X)$ has a basis corresponding to the elements of $F(A, X)$ and hence of power $\leq \bar{X} + \aleph_0$. Since the points of $Tr^\alpha(S)$ are isolated points in S^α , S must have a vanishing perfect kernel. Any compact Hausdorff space with a vanishing perfect kernel has power \leq the power of any basis of it.

The more difficult result is the converse:

THEOREM 3. *The theory Σ is totally transcendental if and only if whenever $S(A, X) \in \mathcal{S}(\Sigma)$ and X is countable then $S(A, X)$ is countable.*

Theorem 3 is the analogue of the Cantor-Bendixson theorem and its proof is similar to proofs of that theorem.

2. The next step is to establish that every theory categorical in an uncountable power satisfies the countability condition of theorem 3 and is therefore totally transcendental. Vaught⁵ has already established this for theories categorical in power \aleph_n . To establish it in general we use the following variant of a result of Ehrenfeucht and Mostowski.

THEOREM 4. *If Σ is a theory having an infinite model and X an arbitrary linearly ordered set, then there is a model A of Σ , and a set $X \subseteq |A|$ such that if $Y_1, Y_2 \subseteq X$ and $f: Y_1 \rightarrow Y_2$ is an order isomorphism then f is an elementary monomorphism of (A, Y_1) to (A, Y_2) .*

By the well known method of replacing existential quantifiers with function symbols we may replace Σ by a new theory Σ' , which has essentially the same models as Σ and such that if A is a model of Σ' and $X \subseteq |A|$ we may define the submodel, $M(X, A)$, generated by X ; indeed, $M(X, A)$ is just the closure of X in A by all the functions.

THEOREM 5. *If Σ is a theory with an infinite model and $K \geq \aleph_0$, then there is a model of Σ , B , such that $\bar{B} = K$ and if $X \subseteq |B|$ and X is countable then only a countable number of points of $S(B, X)$ are realized in B .*

We sketch the proof. Replace Σ by the theory Σ' described above. Let Y be a linearly ordered set having the order type of K (i.e., of the ordinals having power $< K$). Let A be a model of Σ' containing Y and satisfying theorem 4 (with Y in place of X). Then $B = M(A, Y)$ is the required model. (It is necessary to arrange things so that $B < A$, but this can always be done.)

THEOREM 6. *If Σ is categorical in some uncountable power then Σ is totally transcendental.*

Proof: If Σ were not totally transcendental then there would be a model A of Σ , and a countable set $X \subseteq |A|$ such that $S(A, X)$ was not countable. By applying the completeness theorem one easily shows that for each uncountable power, K , there is a model B of Σ , such that $\bar{B} = K$, $X \subseteq |B|$, and B realizes an uncountable

number of points of $S(B, X)$. Then B is certainly not isomorphic to any model satisfying theorem 5 and hence Σ could not be categorical in power K .

3. We define a somewhat weaker notion than saturated. A model A is *countably saturated* if for every countable $X \subseteq |A|$ every point of $S(A, X)$ is realized in A .

THEOREM 7. *If Σ is totally transcendental (and has an infinite model) then for every $K > \aleph_0$ there is a countably saturated model of Σ of power K .*

Proof. Let A_0 be any model of Σ of power K . By Theorem 2, $S(A_0, |A_0|)$ has power K . From the completeness theorem it follows that there is a model, A_1 , such that $A_1 \supseteq A_0$, $\overline{A_1} = K$, and A_1 realizes every point of $S(A_1, |A_0|)$. We iterate this process ω_1 times (taking unions at limit ordinals). Then A_{ω_1} satisfies the theorem. For if $X \subseteq |A_{\omega_1}|$ and X is countable, then there is an $\alpha < \omega_1$ such that $X \subseteq |A_\alpha|$ and hence each point of $S(A_{\omega_1}, X)$ is realized already in $A_{\alpha+1}$.

THEOREM 8. *If Σ is totally transcendental and has an uncountable model which is not saturated, then for every $K \geq \aleph_0$ there is a model of Σ of power K which is not countably saturated.*

Our proof of this theorem is complicated and we shall not give it here. We remark, however, that it involves several transfinite inductions, in which at each step we pick a point of a particular transcendental rank (or often, the point of minimal transcendental rank which satisfies some condition). Thus, we here explicitly use the fact that each $p \in S(\Sigma)$ has a definite transcendental rank. By contrast, in proving theorem 7 we used only the cardinality result of theorem 2.

From Theorems 7 and 8 now follows immediately:

THEOREM 9. *If Σ is categorical in some power $K > \aleph_0$ then every uncountable model of Σ is saturated.*

Theorem 9 and Theorem 1 provide an affirmative answer to the question of Łoś.

¹ Łoś, J., "On the categoricity in power of elementary deduction systems," *Coll. Math.*, **3**, 58–62 (1954); Vaught, R., "Applications of the Löwenheim-Skolem-Tarski theorem to problems of completeness and decidability," *Indag. Math.*, **16**, 467–472 (1954).

² Rull-Nardzewski, C., "On theories categorical in power $\leq \aleph_0$," *Bull. Acad. Polon. Sc.*, Cl. III, **7**, 545–548 (1959); Vaught, R., "Denumerable models of complete theories," *Proceeding of the Symposium on the Foundations of Mathematics: Infinitistic Methods*, 303–321, Warsaw, 1959.

³ Ehrenfeucht, A., and A. Mostowski, "Models of axiomatic theories admitting automorphisms," *Fund. Math.*, **43**, 50–68 (1956); Ehrenfeucht, A., "On theories categorical in power," *Fund. Math.*, **44**, 241–248 (1957).

⁴ Stone, M. H., "Applications of the theory of Boolean rings to general topology," *Trans. Amer. Math. Soc.*, **41**, 321–364 (1937).

⁵ Morley, M., and R. Vaught, "Homogenous universal models," *Math. Scand.*, to appear.

⁶ Mostowski, A., and A. Tarski, "Boolesche Ringe mit geordneter Basis," *Fund. Math.*, **32**, 69–86 (1939).