ACOUSTICAL IMPEDANCE, AND THE THEORY OF HORNS AND OF THE PHONOGRAPH

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The introduction more than thirty years ago of the term 'impedance' by Mr. Oliver Heaviside has been productive of very great convenience in the theory of alternating currents of electricity. Unfortunately, engineers have not seemed to notice that the idea may be made as useful in mechanics and acoustics as in electricity. In fact, in such apparatus as the telephone one may combine the notions of electrical and mechanical impedance with great advantage. Whenever we have permanent vibrations of a single given frequency, which is here denoted, as usual, by $n/2\pi$, the notion of impedance is valuable in replacing all the quantities involved in the reactions of the system by a single complex number. If we follow the convenient practice of denoting an oscillating quantity by e^{int} and taking its real part (as introduced by Cauchy) all the derivatives of e^{int} are obtained by multiplication by powers of in, or graphically by advancing the representative vector by the proper number of right angles.

If we have any oscillating system into which a volume of air X periodically enters under an excess pressure p, I propose to define the impedance by the *complex* ratio $Z = p/X$. If we call $dX/dt = I$ the current as in electricity, if we followed electrical analogy we should write $Z = pI$ so that the definition as given above makes our impedance lead by a right angle the usual definition. I believe this to be more convenient for our purposes than the usual definition and it need cause no confusion.

If we have a vibrating piston of area S as in the phonometer, we shall refer its motion to the volume $S\xi$ it carries with it and the force acting on it to the pressure, so that $F = Sp$. The differential equation of the motion is

$$m\frac{d^2\xi}{dt^2} + \kappa\frac{d\xi}{dt} + f\xi = F = Sp, \qquad X = S\xi, \tag{1}$$

we have

$$Z_1 = (f - mn^2 + i\kappa n)/S^2, \tag{2}$$

where m is the mass, κ the damping, f the stiffness. The real part of S^2Z, $f - mn^2$, is the uncompensated stiffness, which is positive in a system tuned too high, when the displacement lags behind the force, by an angle between zero and one right angle, negative when the system is tuned too low, when the

* This article was read in December 1914 at the meeting of the American Physical Society at Philadelphia, and has been held back because of the continual development of the experimental apparatus described in a previous paper in these Proceedings.

lag is between one and two right angles, as shown in figure 1. If we force air into a chamber of volume V, the compression $s = X/V$ will be related to the excess pressure p by the relation $p = es$, where e is the modulus of elasticity of the air $e = \rho a^2$, ρ being the density and a the velocity of sound. Consequently we have

$$Z_0 = \frac{e}{V} = \frac{\rho a^2}{V}, \tag{3}$$

and the analogy is to a condenser. If we have air passing through an orifice or short tube of conductivity c its inertia gives an apparent mass ρ/c, and if it

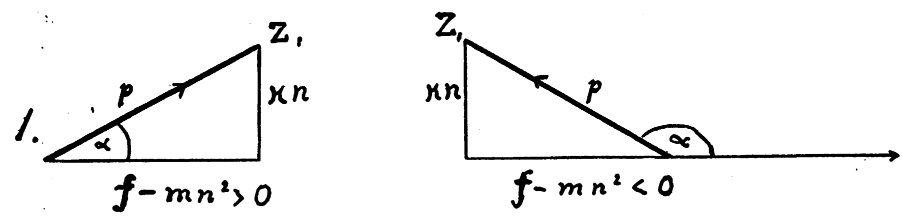

FIG. 1

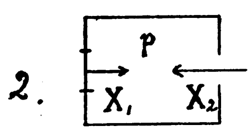

FIG. 2

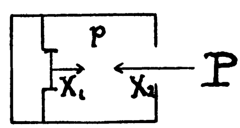

FIG. 3

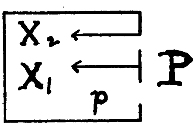

FIG. 4

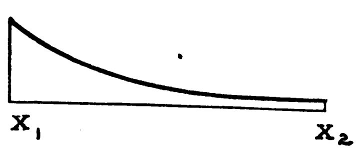

FIG. 5

escapes from a circular hole in an infinite plane it dissipates energy so that the whole impedance credited to the hole is

$$Z_2 = -\frac{\rho n^2}{c} + \frac{\rho n^3}{2\pi a}\, i = ek^2 \left\{\frac{k}{2V}\, i - \frac{1}{c}\right\}, \text{ where } k = \frac{n}{a}. \tag{4}$$

These three typical impedances will be at constant use in acoustics. It is to be remembered that systems in series have their impedances added and in parallel have the reciprocals of impedance added. Also that the free vibrations of a system are obtained by equating the impedances to zero.

As a simple example consider the phone described in the previous article, figure 3.

Let $X_1 = S\xi$ be the volume introduced by the piston X_2 that entering by the hole. Then

$$p = Z_0 (X_1 + X_2) = - Z_2 X_2, \tag{5}$$

$$X_2 = \frac{-Z_0}{Z_0 + Z_2} X_1,$$

and inserting values,

$$X_2 = - \frac{S\xi}{1 + Vk^2\left(\frac{ki}{2\pi} - \frac{1}{c}\right)} \tag{6}$$

Disregarding phase by taking the modulus and putting $k = n/a$ we have the phone formula for the strength of source.

$$A = \left| \frac{dX}{dt} \right| = S\,\Psi\, |\,\xi\,|, \tag{7}$$

where

$$\Psi = \frac{n}{\left\{\left(1 - \frac{Vk^2}{c}\right)^2 + \frac{V^2k^6}{4\pi^2}\right\}^{\frac{1}{2}}} \tag{8}$$

If instead of sending the air out through a hole it goes into a cone or any other horn, we must use for the impedance Z_2 that given below, and we arrive at the theory of the phonograph, and are thus able to answer the question as to the function of the horn in persuading the sound to come out of the phonograph when the motion of the diaphragm is given (it is well known that very little sound is emitted by the phonograph or the telephone with the horn taken off, although in the former case the motion of the diaphragm is exactly the same).

The phonometer was formerly arranged with the back of the diaphragm protected from the sound, figure 3. Let P be the external pressure, then, as before,

$$p = Z_0(X_1 + X_2) \tag{9}$$

and in addition,

$$\begin{aligned} -p &= Z_1 X_1, \\ P - p &= Z_2 X_2, \end{aligned} \tag{10}$$

from which

$$X_1 = \frac{-PZ_0}{Z_0Z_1 + Z_1Z_2 + Z_2Z_0}, \tag{11}$$

giving the formula for the measurement of the pressure,

$$P = \varphi\,\xi/S \tag{12}$$

$$\varphi = \frac{\gamma}{[\{uv - (\alpha\beta + \gamma^2)\}^2 + \{\beta u + \alpha v\}^2]^{\frac{1}{2}}} \tag{13}$$

and φ may be termed the sensitiveness of the phonometer. Where

$$\gamma = S^2 Z_0 = S^2 \rho a^2/V, \qquad \alpha = \kappa n, \qquad \beta = S^2 \rho n^3/2\pi a,$$

$$u = f - n^2 m + S^2 \rho a^2/V, \qquad v = S^2\{\rho a^2/V - \rho u^2/c\} = S^2 Z_0 (1 - k^2 V/c) \quad (14)$$

As described in my recent article the back of the piston is exposed to the sound, figure 4. Then

$$\begin{aligned} P - p &= Z_1 X_1 = Z_2 X_2 \\ p &= Z_0 (X_1 + X_2) \end{aligned} \qquad (15)$$

from which

$$X_1 = \frac{Z_2 P}{Z_0 Z_1 + Z_1 Z_2 + Z_2 Z_0} \qquad (16)$$

$$\varphi = \left[\frac{(v-\gamma)^2 + \beta^2}{\{uv - (\alpha\beta + \gamma^2)\}^2 + (\alpha v + \beta u)^2}\right]^{\frac{1}{2}}, \qquad (17)$$

Tubes and Horns.—Beside the above described phone and phonometer, the theory of which assumed a resonator so small that the pressure is supposed to be the same at every internal point, I have made use of many arrangements employing tubes or cones, in which we must take account of wave-motion. The familiar theory of cylindrical pipes may be included in the following generalized theory, which I have found experimentally to serve well.

Let us consider a tube of infinitesimal cross section σ varying as a function of the distance x from the end of the tube. Then if q is the displacement of the air, p the pressure, s the compression, we have the fundamental equations

$$p = es = \rho a^2 s = - e \operatorname{div} q = -\frac{e}{\sigma}\frac{d(q\sigma)}{dx}, \qquad (18)$$

$$\frac{d^2 p}{dt^2} = a^2 \Delta p = a^2 \operatorname{div}\operatorname{grad} p = a^2\left\{\frac{1}{\sigma}\frac{d}{dx}\left(\sigma\frac{dp}{dx}\right)\right\} \qquad (19)$$

$$\frac{d^2 q}{dt^2} = -\frac{1}{\rho}\frac{dp}{dx} = a^2 \frac{d}{dx}\left\{\frac{1}{\sigma}\frac{d}{dx}(q\sigma)\right\}. \qquad (20)$$

For a simple periodic motion we put p, q proportional to e^{int}, and obtain

$$\frac{d^2 p}{dx^2} + \frac{d\log\sigma}{dx}\frac{dp}{dx} + k^2 p = 0, \quad \frac{d^2 q}{dt^2} + \frac{d\log\sigma}{dx}\frac{dq}{dx} + \frac{d^2\log\sigma}{dx^2} q + k^2 q = 0. \quad (21)$$

Both these linear equations may be solved by means of series, and if we call $u(kx)$, $v(kx)$ two independent solutions we have

$$p = Au + Bv, \quad \beta q = Au' + Bv', \quad \beta = \rho a^2 k,$$

where the accents signify differentiation according to kx. If we denote values at one end $x = x_1$ and at the other end $x = x_2$ by suffixes 1, 2, respectively, and form the determinants

$$D_1 = \begin{vmatrix} u_1, v_1 \\ u_1', v_1' \end{vmatrix}, \quad D_2 = \begin{vmatrix} u_2, v_2 \\ u_2', v_2' \end{vmatrix}, \quad D_3 = \begin{vmatrix} u_1, v_1 \\ u_2', v_2' \end{vmatrix},$$
$$D_4 = \begin{vmatrix} u_2, v_2 \\ u_1', v_1' \end{vmatrix}, \quad D_5 = \begin{vmatrix} u_1, v_1 \\ u_2, v_2 \end{vmatrix}, \quad D_6 = \begin{vmatrix} u_1', v_1' \\ u_2', v_2' \end{vmatrix}, \tag{22}$$

which satisfy the relation,

$$D_1D_2 = D_3D_4 + D_5D_6$$

we may determine the constants A, B in terms of any two out of p_1, q_1, p_2, q_2, so that we obtain

$$\begin{aligned} p_2 &= (p_1D_4 + \beta q_1D_5)/D_1, & \beta q_2 &= (-p_1D_6 + \beta q_1D_3)/D_1, \\ p_1 &= (p_2D_3 - \beta q_2D_5)/D_2, & \beta q_1 &= (p_2D_6 + \beta q_2D_4)/D_2. \end{aligned} \tag{23}$$

As it is more convenient to deal with the volumes $X_1 = \sigma_1 q_1$, $X_2 = \sigma_2 q_2$ we shall have in general

$$p_2 = ap_1 + bX_1, \quad X_2 = cp_1 + dX_1, \tag{24}$$

where

$$a = \frac{D_4}{D_1}, \quad b = \frac{\beta}{\sigma_1}\frac{D_5}{D_1} \quad c = -\frac{\sigma_2}{\beta}\frac{D_6}{D_1}, \quad d = \frac{\sigma_2}{\sigma_1}\frac{D_3}{D_1}, \quad ad - bc = \frac{\sigma_2}{\sigma_1}\frac{D_2}{D_1},$$

and for the impedances belonging to the ends of the tube

$$Z_2 = \frac{aZ_1 + b}{cZ_1 + d_1}, \qquad Z_1 = \frac{dZ_2 - b}{-cZ_2 + a}, \tag{25}$$

so that the impedance at either end of the tube is a linear fractional function of the other. According to the apparatus attached to an end the impedance attached to that end is known. A tube for which a, b, c, d are given may be replaced by any other tube having the same constants.

Examples.—Cylindrical tube, σ constant. Put $x_2 - x_1 = Z_1$,

$$\frac{d^2p}{dx^2} + k^2p = 0,$$

$$u = \cos kx, \qquad r = \sin kx, \qquad u' = -\sin kx, \qquad v' = \cos kx, \tag{26}$$

$$D_1 = D_2 = 1, \qquad D_3 = D_4 = \cos kl, \qquad D_5 = D_6 = \sin kl, \tag{27}$$

$$a = d = \cos kl, \qquad b = \frac{\beta}{\sigma}\sin kl, \qquad c = -\frac{\sigma}{\beta}\sin kl,$$

$$Z_2 = \frac{\beta}{\sigma}\,\frac{Z_1\cos kl + \frac{\beta}{\sigma}\sin kl}{-Z_1\sin kl + \frac{\beta}{\sigma}\cos kl,}, \quad Z_1 = \frac{\beta}{\sigma}\,\frac{Z_2\cos kl - \frac{\beta}{\sigma}\sin kl,}{Z_2\sin kl + \frac{\beta}{\sigma}\cos kl} \tag{28}$$

Conical tube, $\sigma = \sigma_0 x^2$

$$\frac{d^2p}{dx^2} + \frac{2}{x}\frac{dp}{dx} + k^2x = 0 \tag{29}$$

$$u = \frac{\cos kx}{kx}, \quad v = \frac{\sin kx}{kx}, \quad u' = -\left(\frac{\sin kx}{kx} + \frac{\cos kx}{k^2x^2}\right), \quad v' = \frac{\cos kx}{kx} - \frac{\sin kx}{k^2x^2}$$

$$D_1 = \frac{1}{k^2{x_1}^2}, \quad D_2 = \frac{1}{k^2{x_2}^2}, \quad D_2 = \frac{\cos kl}{k^2x_1x_2},$$

$$D_4 = \frac{\cos kl}{k^2x_1x_2} + \frac{\sin kl}{k^3x_1{x_2}^2}, \quad D_5 = \frac{\sin kl}{k^2x_1x_2}$$

and if we introduce two lengths ϵ_1, ϵ_2, defined by the equations

$$\tan k\epsilon_1 = kx_1, \qquad \tan k\epsilon_2 = kx_2,$$

we easily get

$$a = \frac{x_1}{x_2}\frac{\sin k(l + \epsilon_1)}{\sin k\epsilon_1}, \qquad b = \frac{\beta}{\sigma_1}\frac{x_1}{x_2}\sin kl, \tag{30}$$

$$c = -\frac{\sigma_2}{\beta}\frac{x_1}{x_2}\frac{\sin k(l + \epsilon_1 - \epsilon_2)}{\sin k\epsilon_1 \sin k\epsilon_2}, \qquad d = \frac{\sigma_2}{\sigma_1}\frac{x_1}{x_2}\frac{\sin k(l - \epsilon_2)}{\sin k\epsilon_2}, \tag{31}$$

$$Z_2 = -\frac{\beta}{\sigma_2}\frac{Z_1\dfrac{\sin k(l + \epsilon_1)}{\sin k\epsilon_1} + \dfrac{\beta}{\sigma_1}\sin kl,}{Z_1\dfrac{\sin k(l + \epsilon_1 - \epsilon_2)}{\sin k\epsilon_1 \sin k\epsilon_2} + \dfrac{\beta}{\sigma_1}\dfrac{\sin k(l - \epsilon_2)}{\sin k\epsilon_2}}$$

$$Z_1 \doteq -\frac{\beta}{\sigma_1}\frac{Z_2\dfrac{\sin k(l - \epsilon_1)}{\sin k\epsilon_2} + \dfrac{\beta}{\sigma_2}\sin kl}{Z_2\dfrac{\sin k(l + \epsilon_1 - \epsilon_2)}{\sin k\epsilon_1 \sin k\epsilon_2} + \dfrac{\beta}{\sigma_2}\dfrac{\sin k(l + \epsilon_1)}{\sin k\epsilon_2}} \tag{32}$$

The formulae (31), (32) were used by Professor G. W. Stewart in designing horns to be used during the war.

It is not true, as is frequently stated in books on musical instruments, that the brass instruments of the orchestra are hyperbolic in profile, but I have found for all practical purposes the bell of every instrument may be represented by one of the three formulae

$$\sigma = \sigma_0 x^n, \qquad \sigma = \sigma_0 e^{-mx}, \qquad \sigma = \sigma_0 e^{-mx^2}$$

Even if an equation cannot be given to the profile the differential equation may be easily integrated graphically, or the length may be divided up into sections and different values of n used for different sections, as is customary in the theory of ballistics.

Case 1. $\sigma = \sigma_0 x^n$. (Change units so that $k = 1$)

$$\frac{d^2p}{dx^2} + \frac{n}{x}\frac{dp}{dx} + p = 0, \qquad \frac{d^2X}{dx^2} - \frac{n}{x}\frac{dX}{dx} + X = 0 \tag{34}$$

We have

$$p = J_{\pm\frac{n-1}{2}}(x)/x^{\frac{n-1}{2}}, \qquad X = J_{\pm\frac{n+1}{2}}(x)x^{\frac{n+1}{2}}, \tag{35}$$

Examples.

$$n = 0, \qquad n = 2, \qquad n = -2$$

$$J_{\frac{1}{2}}(x) = \sin x/\sqrt{x}, \qquad J_{\frac{3}{2}}(x) = \sin x/x^{\frac{3}{2}} - \cos /x\sqrt{x},$$

$$J_{-\frac{1}{2}}(x) = \cos x/\sqrt{x}, \qquad J_{-\frac{3}{2}}(x) = -\sin x/\sqrt{x} - \cos x/x^{\frac{3}{2}}$$

These include the straight cylinder, the straight cone, and the purely hyperbolic horn. In the latter case we have figure 5, where x_1, is the bell. If the horn is closed at x_2 we have

$$Z_2 = \infty$$

$$Z_1 = ck^2\left\{\frac{1}{c} - \frac{k}{2\pi}i\right\} = -\frac{d}{c} = \frac{(\sin kl + kx_1 \cos kl)\,\beta x_1}{\sigma_0 k \sin kl_1}$$

and if we put $\xi = kl$

$$\operatorname{ctn}\,\xi = \frac{\sigma_1}{lc}\xi - \frac{l}{x_1}\frac{1}{\xi}, \tag{36}$$

which may be easily discussed graphically.

On the other hand if the horn is open at x_2 we have

$$\tan\xi/\xi = \left(1 - \frac{\sigma x_1 x_2}{cl^3}\xi^2\right)\Big/\left(1 + \xi^2\left\{\frac{x_1x_2}{l^2} - \frac{\sigma_1 x_1}{cl^2}\right\}\right). \tag{37}$$

These formulae were confirmed experimentally by my then assistant Dr. H. K. Stimson in 1915 on a coach-horn, a trombone, and a phonograph horn, with the following results:

		CALCULATED	OBSERVED
For the coach-horn	Closed	177	181
	Open	254	202
For the trombone	Closed	286	305
	Open	418	432
For the phonograph	Closed	311	304
	Open	329	415

These results give a fair agreement considering that we have used for the conductivity of the mouth the simple formula $c = 0.6\,R$ which is true only for cross-sections infinitesimal compared with the wave-length, whereas in the case of the wooden phonograph horn, the actual radius is nearly one-fourth of the wave-length.

A paper on the subject of the impedance of such an end will shortly appear.

In the case of an exponential section we have

$$\sigma = \sigma_0 e^{-mx}$$

$$\frac{d^2p}{dx^2} + m\frac{dp}{dx} + p = 0, \qquad \frac{d^2X}{dx^2} - m\frac{dX}{dx} + X = 0,$$

$$p = e^{-\sqrt{4-k^2}x}\{A\cos kx + B\sin kx\},$$

$$X = e^{-\sqrt{4-k^2}x}\{C\cos kx + D\sin kx\}.$$

and it is noticeable that the pressure vanishes at the same cross-section as for a straight tube.

Finally, in the case

$$\sigma = \sigma_0 e^{-mx^2}$$

we may solve the equation by means of the confluent hyper-geometric function.

It is to be noticed that in none of these cases, except the straight tube, are the different overtones harmonic. Thus, the characteristic tone of the "brass" is not due to the substance, but is entirely a matter of geometry as is shown by the heavy casting in plaster of Paris of a trombone bell used by the writer, the tone of which cannot be distinguished from that of the brass bell. I believe this phenomenon is well known.

Inasmuch as all musical instruments are composed either of resonators combined with strings, bars, plates, and horns, I feel that the above theory, while merely an approximation as to accuracy, will go far toward enabling us to complete the theory of musical instruments. Of course, the actual tones emitted by a brass instrument will depend upon the dynamics of the lips which is reserved for a future paper.