*FUNCTORIAL SEMANTICS OF ALGEBRAIC THEORIES**

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When they introduced the theory of categories in 1945,[1] Eilenberg and Mac Lane suggested the possibility of "functorizing" the study of general algebraic systems. The author has carried out the first steps of this program, making extensive use of the theory of adjoint functors, as introduced by Kan[3] and refined by Freyd.[2] In some directions a very great degree of useful generality can be achieved; for example, the constructions of free algebras, tensor algebras, monoid rings, enveloping algebras of Lie algebras, abelianization of groups, and covariant extension of rings for modules can all be viewed in a unified way as adjoints to "algebraic" functors, and we show that such adjoints *always* exist. Also, by formalizing "semantics" itself as a functor and showing that it has an adjoint, we are able to give a new characterization of equational classes of algebras (viewed as abstract categories) and also to provide a canonical tool for the partial analysis of many nonalgebraic categories and functors.

By an *algebraic theory* we mean a small category **A** whose objects are the natural numbers 0, 1, 2, ... and in which each object n is the categorical direct product of the object 1 with itself n times. By an *n-ary operation* of **A** is meant any map $n \to 1$ in **A**. Since n is a product, the projections $\pi_i^{(n)}$: $n \to 1, i = 0, 1, \ldots, n - 1$ are always n-ary operations for each n in any algebraic theory, but in general there will be more. The maps $n \to m$ in an algebraic theory **A** are in one-to-one correspondence with the m-tuples of n-ary operations of **A**. Any "presentation" of a concept of algebraic structure (e.g., groups, modules over a given ring, Jordan algebras, lattices, etc.), which involves a set of symbols denoting finitary operations together with a set of equations (= identities) relating composite operations, determines an algebraic theory, and conversely every algebraic theory has such presentations. By a mapping between algebraic theories we will understand a functor which preserves products and takes 1 into 1. Algebraic theories and the mappings between them thus form a category $\mathcal{T}$.

Each algebraic theory **A** determines a large category $\mathcal{S}^{(\mathbf{A})}$ whose class of objects is just the equational class (variety) of all algebras of type **A**, and whose maps are all (into) homomorphisms between these. An algebra of type **A** can be viewed as a product preserving functor $\mathbf{A} \to \mathcal{S}$ from **A** to the category of sets; a homomorphism of algebras is then just a natural transformation between such functors. If **A** is the algebraic theory whose *only* n-ary operations are projections (i.e., **A** is equivalent to the dual of the category of finite sets), then the category of algebras $\mathcal{S}^{(\mathbf{A})}$ is just the category $\mathcal{S}$ of sets. Every map f: $\mathbf{A} \to \mathbf{B}$ of algebraic theories determines in an obvious way a functor $\mathcal{S}^{(f)}$: $\mathcal{S}^{(\mathbf{B})} \to \mathcal{S}^{(\mathbf{A})}$ which preserves underlying sets, i.e., for which $U_{\mathbf{B}} = \mathcal{S}^{(f)} U_{\mathbf{A}}$, where $U_{\mathbf{A}}$: $\mathcal{S}^{(\mathbf{A})} \to \mathcal{S}$, $U_{\mathbf{B}}$: $\mathcal{S}^{(\mathbf{B})} \to \mathcal{S}$ are the underlying set functors (notice the order in which we write composition). We call any functor of the form $\mathcal{S}^{(f)}$ an *algebraic functor*, and we call any category of the form $\mathcal{S}^{(\mathbf{A})}$ an *algebraic category*. Any algebraic theory **A** is equivalent to the dual of the full category of finitely generated free algebras in its associated algebraic category.

THEOREM. *Every algebraic functor has an adjoint.*

For example, the category of rings (with unit) and that of monoids are algebraic, and the functor which assigns to each ring the monoid consisting of the same elements under multiplication alone is an algebraic functor. The adjoint to this functor is the well-known construction of the monoid ring. Other instances of this theorem were mentioned in the first paragraph. Note that for $f\colon \mathbf{A} \to \mathbf{B}$ in $\mathfrak{T}$, the natural transformation from the identity functor on $\mathcal{S}^{(\mathbf{A})}$ to $F\mathcal{S}^{(f)}$, where F is the adjoint of $\mathcal{S},^{(f)}$ need not be a monomorphism (for example, not every Jordan algebra is special—on the other hand every distributive lattice can be embedded in a Boolean ring); a description of those f for which this is so would provide a unified solution to a great number of problems in algebra—for example, when can an algebra be enlarged to contain a root of a given equation?

If we write $\mathbf{A}\mathfrak{S} = U_{\mathbf{A}}$, $f\mathfrak{S} = \mathcal{S}^{(f)}$, we obtain a functor $\mathfrak{S}$ which we call *algebraic semantics;* the domain of $\mathfrak{S}$ is the dual $\mathfrak{T}^*$ of the category of algebraic theories, and we take as its codomain the category $\mathcal{K}$ whose objects are functors $U\colon\mathfrak{X} \to \mathcal{S}$ with arbitrary domain category and with the category of sets as codomain, subject only to the restriction that for each natural number n, the class of all natural transformations $U^n \to U$ is small, where U^n assigns to each X the nth Cartesian power of $XU;$ a map $T\colon U \to U'$ in $\mathcal{K}$ is to be any functor $T\colon\mathfrak{X} \to \mathfrak{X}'$ for which $U = TU'$.

THEOREM. *Algebraic semantics has an adjoint $\hat{\mathfrak{S}}\colon\mathcal{K} \to \mathfrak{T}^*$ (which we call algebraic structure), and furthermore $\mathfrak{S}\hat{\mathfrak{S}}$ is naturally equivalent to the identity functor of $\mathfrak{T}^*$. Explicitly, for any U in $\mathcal{K}$ the n-ary operations of the algebraic theory $U\hat{\mathfrak{S}}$ are the natural transformations $U^n \to U$.*

Thus, any category $\mathfrak{X}$ equipped with an "underlying set functor" U determines an algebraic category $\mathcal{S}^{(U\hat{\mathfrak{S}})}$ together with a functor $\Phi\colon\mathfrak{X} \to \mathcal{S}^{(U\hat{\mathfrak{S}})}$ which preserves underlying sets, and given any other such functor $\Psi\colon\mathfrak{X} \to \mathcal{S}^{(\mathbf{A})}$, there is a unique $f\colon\mathbf{A} \to U\hat{\mathfrak{S}}$ in $\mathfrak{T}$ such that $\Psi = \Phi\mathcal{S}^{(f)}$. Also, the operations which define an algebraic category are in natural one-to-one correspondence with the natural operations on its underlying set functor. Thus the functor $\hat{\mathfrak{S}}$ becomes most interesting when applied to "underlying set functors" on nonalgebraic categories. For example, if we take for $\mathfrak{X}$ the dual of the category of sets and for U the (contravariant) power set functor, the algebraic structure of U is the theory of Boolean algebras and Φ assigns to each set the Boolean algebra of its subsets. Again, if we take U as the functor which assigns to every group G the set of all integer-valued functions on G with finite support, and to every group homomorphism $g\colon G \to G'$ the function $gU\colon GU \to G'U$ defined by $(x')(f)(gU) = \Sigma\{xf|xg = x'\}$ for $f \in GU$, $x' \in G'$, then $U\hat{\mathfrak{S}}$ is an extension of the theory of rings which includes two additional unary operations, "involution" and "trace."

The above theorem implies that a category $\mathfrak{X}$ is equivalent to some algebraic category *iff* it has some underlying set functor U such that the particular functor $\Phi\colon\mathfrak{X} \to \mathcal{S}^{(\mathbf{A})}$ described above is an equivalence where $\mathbf{A} = U\hat{\mathfrak{S}}$. In that case we must actually have $U = \mathrm{Hom}(G, ?)$ where G is an object in $\mathfrak{X}$ such that $G\Phi$ is a free $\mathbf{A}$-algebra on one generator. These observations enable us to completely characterize algebraic categories in the theorem below. We first define some terms.

By an *equalizer* and *coequalizer* of a pair of maps

$$X \overset{f_0}{\rightrightarrows} Y$$

in a category $\mathfrak{X}$, we mean respectively an inverse limit $j: K \rightarrow X$ and a direct limit $p: Y \rightarrow K^*$ of the above diagram in the sense of Kan.[3] A map p is called a *regular epimap iff* it is a coequalizer of some pair of maps. A category *has finite limits iff* it has an initial and a final object and if all possible binary products and coproducts, as well as all possible equalizers and coequalizers, exist. A pair f_0, f_1 as above is called a *precongruence iff* the corresponding $f: X \rightarrow Y \times Y$ is monic as well as reflexive, symmetric, and transitive in an obvious sense; the pair is called a *congruence iff* f is the equalizer of the pair $<\pi_0 p, \pi_1 p>$ where $\pi_i: Y \times Y \rightarrow Y$ are the projections and p is the coequalizer of $<f_0, f_1>$. In a category with finite limits, every congruence is a precongruence, while in any algebraic category the converse is also true.

An object G in a category $\mathfrak{X}$ is *abstractly finite iff* the following three conditions hold: (*a*) for every object X, the class $\text{Hom}(G, X)$ of maps $G \rightarrow X$ is small (i.e., is a "set"); (*b*) for every small class I, the I-fold coproduct (= free product) $I \cdot G$ exists; (*c*) any map $G \rightarrow I \cdot G$ factors through some $I' \cdot G$ where $I' \subset I$ and I' is finite. A map $f: Y \rightarrow Z$ in $\mathfrak{X}$ is *G-surjective iff* every map $G \rightarrow Z$ factors across f. An object G is a *regular projective generator iff* the G-surjective maps are precisely the regular epimaps.

THEOREM. *Let $\mathfrak{X}$ be a category with the following properties: (0) $\mathfrak{X}$ has finite limits, (1) $\mathfrak{X}$ has an abstractly finite regular projective generator G. Then there is an algebraic theory* $\mathbf{A}$ *and a functor $\Phi: \mathfrak{X} \rightarrow \mathcal{S}^{(\mathbf{A})}$ which is full, faithful, and has an adjoint, and the free objects in $\mathfrak{X}$ coincide with those in $\mathcal{S}^{(\mathbf{A})}$ (i.e., $(I \cdot G)\Phi = I \cdot (G\Phi)$). Furthermore, Φ is an equivalence iff (2) every precongruence in $\mathfrak{X}$ is a congruence. Conditions 0, 1, 2 are necessary and sufficient that $\mathfrak{X}$ be equivalent to some algebraic category.*

This theorem is closely related to an unpublished theorem characterizing "quasi-primitive" categories of algebras which was found by J. R. Isbell, who also pointed out that my original proof of the above could be simplified by noting that under conditions 0 and 1, every subalgebra of $X\Phi$, for X in $\mathfrak{X}$, is itself of the form $X'\Phi$. A category satisfying 0 and 1, but not 2, is that of all torsion-free abelian groups; the adjoint to Φ in this case consists of dividing by the torsion subgroup.

COROLLARY. *If $\mathfrak{X}$ is an algebraic category and if $\mathcal{C}$ is any small category with finitely many objects, then the full category $\mathfrak{X}^{\mathcal{C}}$ of functors $\mathcal{C} \rightarrow \mathfrak{X}$ and natural transformations thereof is itself equivalent to an algebraic category.*

The proof of this corollary involves interpreting the underlying set of a functor to mean the product of the underlying sets of the objects involved. When $\mathcal{C}$ has exactly one object, then $\mathcal{C}$ is a monoid and the corollary refers to the well-known notion of a monoid acting by endomorphisms of an algebra to form a new algebra.

COROLLARY. *An abelian category is algebraic iff it is the category of all modules over some associative ring with unity.*

This follows immediately from a theorem of Freyd;[2] condition 2 of the theorem above is always satisfied in an abelian category. The nature of the rings whose existence is implied by these two corollaries is being investigated.[4]

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[1] Eilenberg, S., and S. Mac Lane, "General theory of natural equivalences," *Trans. Amer. Math. Soc.*, **58**, 231–294 (1945).

[2] Freyd, P. J., thesis: Functor Theory, Princeton University, 1960.
[3] Kan, D. M., "Adjoint functors," *Trans. Amer. Math. Soc.*, **87**, 294–329 (1958).
[4] Lawvere, F. W., "The convolution ring of a small category," *Notices Amer. Math. Soc.*, **10**, 280 (1963); Errata, *Notices Amer. Math. Soc.*, **10**, 516 (1963).

*THE CRYSTAL STRUCTURE OF AN INTERMOLECULAR NUCLEOSIDE COMPLEX: ADENOSINE AND 5-BROMOURIDINE**

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The concept of hydrogen-bonding specificity between the purine and pyrimidine bases, adenine and thymine, guanine and cytosine, is fundamental in the present theory of nucleic acid structure and replication. The base-pairing scheme proposed by Watson and Crick[1] in their structural hypothesis for the DNA molecule has gained wide acceptance among biologists and recently has received strong support from three-dimensional Fourier analysis of DNA fiber X-ray diffraction data,[2] and from structure analysis of single crystals containing guanine and cytosine derivatives in an intermolecular complex.[3, 4] On the other hand, Hoogsteen[5] has found a different base-pairing configuration in a crystalline complex of 9-methyl adenine and 1-methyl thymine. Here, the ring nitrogen N3 of thymine hydrogen-bonds to the imidazole nitrogen N7 of adenine instead of bonding to N1 as in the Watson-Crick model. A similar pairing configuration has recently been found in a crystalline complex containing 9-ethyl adenine and 1-methyl uracil.[6] This pairing is of considerable interest since it is thought to occur in the triple-stranded 2:1 complex of polyuridylic acid and polyadenylic acid.[7]

The information derived from these structure investigations has prompted us to investigate other possibilities for cocrystallization of important compounds known to interact in biological systems. The present work describes a single crystal analysis of a nucleoside intermolecular complex between adenosine and 5-bromouridine. The presence of the sugars on the purine and pyrimidine bases brings this model system close to the biological systems of interest. Furthermore, the bromine-substituted derivative is of particular interest since the closely related molecule bromodeoxyuridine is a well-known mutagenic agent. The results show the existence of a third type of base-pair configuration.

Methods.—The adenosine-5-bromouridine complex was crystallized in clusters of thin needles by slow evaporation from an aqueous solution containing equimolar quantities of these compounds. Ultraviolet absorption measurements on aqueous solutions made from single crystals confirmed the presence of the two nucleosides in approximately equal proportions. The crystals were found to be orthorhombic, space group $P22_12_1$ with $a = 4.80 \pm 0.01$, $b = 15.19 \pm 0.01$, and $c = 31.76 \pm 0.03$ Å; the density determined in benzene-methyliodide solutions was 1.706 ± 0.010 gm/cc. The unit cell contains four asymmetric units, each consisting of an adenosine-bromouridine pair and a water molecule. Equi-inclination Weissenberg photographs were taken about the a axis with filtered CuKα radiation, using the multiple film technique. The intensities were estimated visually and corrected with the appropriate Lorentz-polarization factors. A total of 2,511 reflections were indexed, of which 2,015 were nonzero, representing about 90 per cent of the data accessible in the copper sphere. No correction was made for absorption effects.