The solvolysis rate constants of IV and the undeuterated analogue were measured in 50 per cent ethanol at 50°. The results were $k_H = 8.30 \times 10^{-4} \cdot \text{sec}^{-1}$ and $k_D = 7.58 \times 10^{-4} \cdot \text{sec}^{-1}$ ($k_H/k_D = 1.09$). In Table 2 the observed isotope effect is compared to those reported for solvolyses of some other tertiary halides. It can be seen that in our case the rate retardation caused by three $\beta$-deuteriums (9%) is indeed much smaller than generally observed (40–50%).

Work is in progress to test the scope of the latter observation and the validity of the interpretation.

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10 In a recent article D. D. Roberts [J. Org. Chem., 29, 294 (1964)] found only a fourfold rate acceleration in acetylolation of (1-methylcyclopropyl)-carbinyl tosylate relative to cyclopropylcarbinyl tosylate, in agreement with our results.

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SURPLUS FREE POLES OF APPROXIMATING RATIONAL FUNCTIONS*

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Under broad conditions, as expressed in Theorem 1 below, rational functions which approximate on a point set $E$ of the $z$ plane to a function analytic on $E$ and meromorphic in a region containing $E$ have free poles which approach the respective poles of $f(z)$. However, if the rational functions have an excess of free poles, the question arises as to the asymptotic behavior of those poles, and the possible effect on degree of convergence on $E$ and elsewhere if those poles are suppressed.

The present note makes a modest contribution to this study, in both general theory and by specific examples.
To be more explicit, we envisage a point set $E$ which is closed and bounded, whose complement $K$ is regular in the sense that it possesses Green's function $G(z)$ with pole at infinity. We denote generically by $\Gamma_\sigma$ the locus $G(z) = \log \sigma (\geq 0)$ in $K$, and by $E_\sigma$ the (open) interior of $\Gamma_\sigma$; we set $\Phi(z) \equiv \exp(G(z))$.

A rational function of the form
\begin{equation}
a_0z^l + a_1z^{l-1} + \ldots + a_j \\
b_0z^k + b_1z^{k-1} + \ldots + b_k
\end{equation}
where the denominator does not vanish identically is said to be of type $(j,k)$. Then we have\(^1\). \(^2\)

**Theorem 1.** Let $E$ be a closed bounded set whose complement is connected and regular, let $f(z)$ be analytic on $E$, and let $\rho (\leq \infty)$ be the largest number such that $f(z)$ is meromorphic with precisely $\nu(\geq 0)$ poles in $E_\rho$. Suppose the rational functions $R_n(z)$ of respective types $(n,\nu)$ satisfy for the Tchebycheff (uniform) norm on $E$
\[ \lim_{n \to \infty} \sup \| f(z) - R_n(z) \|^{1/n} \leq 1/\rho. \] (2)

Let $D$ denote $E_\rho$ with the $\nu$ poles of $f(z)$ deleted. Then for $n$ sufficiently large the function $R_n(z)$ has precisely $\nu$ finite poles, which approach respectively the $\nu$ poles of $f(z)$ in $E_\rho$. The functions $R_n(z)$ converge to $f(z)$ throughout $D$. For any closed bounded continuum $S$ in $D$ and in the closed interior of $E_\rho, 1 < \sigma < \rho$, we have
\[ \lim_{n \to \infty} \sup \{ \max |f(z) - R_n(z)|, z \text{ on } S \}^{1/n} \leq \sigma/\rho. \] (3)

Hitherto the sequence $R_n(z)$ need not be defined for every $n$, but below the $R_n(z)$ shall be so defined.

The equality sign holds in (2), and in (3) if $S$ contains a point of $\Gamma_\sigma$; moreover, (2) is valid for the functions $R_n(z)$ of types $(n,\nu)$ of best approximation to $f(z)$ on $E$.

The primary topic of the present note is indicated in

**Theorem 2.** Let $f(z)$ in Theorem 1 have either a singularity other than a pole or at least $\mu - \nu + 1$ poles on $\Gamma_\mu, \mu > \nu$. Let $R_n(z)$ be a sequence of rational functions of type $(n,\mu)$ not necessarily defined for every $n$, satisfying
\[ \lim_{n \to \infty} \sup \| f(z) - R_n(z) \|^{1/n} \leq 1/\rho. \] (4)

Let each $R_n(z)$ have precisely one simple pole at $z = \alpha_n$ approaching the point $\alpha$ in $E_\rho - E$, where $f(z)$ is analytic at $z = \alpha$. If $R_n(z)$ denotes the principal part of the pole of $R_n(z)$ at $\alpha_n$, we have
\[ \lim_{n \to \infty} \sup \{ \max |R_n(z) |, z \text{ on } E_0 \}^{1/n} \leq \Phi(\alpha)/\rho, \] (5)
\[ \lim_{n \to \infty} \sup \{ f(z) - |R_n(z) - R_n(z) | \}^{1/n} \leq \Phi(\alpha)/\rho, \] (6)
where $E_0$ is any closed bounded set not containing $\alpha$.

If the finite poles of the $R_n(z)$ are not bounded in their totality, Theorem 2 can obviously be applied to any subsequence of the $R_n(z)$ whose finite poles are bounded in their totality, and can also be applied to any subsequence of the $R_n(z)$, some of whose poles $\beta_\lambda$ (say, $\lambda$ in number) become infinite. In the latter case we simply replace each $R_n(z)$ of the subsequence by $R_n(z)$ multiplied by the factor

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\(^1\) Reference

\(^2\) Reference
\[
\lambda' \prod_{j=1}^{\lambda'} \frac{z - \beta_j}{-\beta_j}, \quad \lambda' \leq \lambda,
\]
whenever \(|\beta_j|\) exceeds a number \(B\), where both \(E_\rho\) and all the \(\mu - \lambda\) poles of \(R_{n\mu}(z)\) which are bounded lie in \(|z| \leq B\); this replacement does not alter the hypothesis (4) nor the conclusion (5) and (6).

We assume henceforth that the finite poles of the \(R_{n\mu}(z)\) are bounded in their totality.

It is a consequence of the method of proof of Theorem 1 that each pole of \(f(z)\) in \(E_\rho\) is the limit of poles of \(R_{n\mu}(z)\) of at least the same total multiplicity, so at most \(\mu - \nu\) finite poles of the original \(R_{n\mu}(z)\) can become infinite.

A chief difficulty in the proof of Theorem 2 is that the poles of the \(R_{n\mu}(z)\) may lie everywhere dense in a subregion of \(E_\rho\) - \(E\), or on a curve in \(E_\rho - E\) (compare ref. 2, Theorem 5). The function which is the limit of the sequence \(R_{n\mu}(z)\) might conceivably then not be a monogenic analytic function even in two disjoint regions of convergence not separated by \(\Gamma_\rho\). This eventuality cannot arise in the situation of Theorem 1, where the \(\nu\) finite poles of the \(R_{n\mu}(z)\) approach the \(\nu\) poles of \(f(z)\) in \(E_\rho - E\), so the limit points of the poles of the \(R_{n\mu}(z)\) cannot separate the plane.

Given a neighborhood of \(\alpha\) of Theorem 2 containing no pole of \(f(z)\) nor point of \(E_0\), and lying in some \(E_\sigma\), \(\sigma < \rho\), we choose in that neighborhood a circle \(\gamma\) with center \(\alpha\), containing on or within it no limit point of the poles other than \(\alpha\) of the \(R_{n\mu}(z)\). By use of the corresponding function of Theorem 1 we have from (2) and (4)

\[
\lim_{n \to \infty} \sup_{z} |R_{n\mu}(z) - R_{n\sigma}(z)|^{1/n} \leq 1/\rho.
\]

Consequently, by slight modification of Lemma 3 of reference 1, we have

\[
\lim_{n \to \infty} \sup_{z, \gamma} |R_{n\mu}(z) - R_{n\sigma}(z)|^{1/n} \leq \sigma/\rho.
\]

By (3), we may write from (8)

\[
\lim_{n \to \infty} \sup_{z, \gamma} |f(z) - R_{n\mu}(z)|^{1/n} \leq \sigma/\rho.
\]

We have for \(z\) on \(E_0\) for \(n\) sufficiently large

\[
-R_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(t) - R_{n\mu}(t)}{t - z} \, dt.
\]

By (9) we deduce (5) with the second member replaced by \(\sigma/\rho\), and letting \(\sigma\) approach \(\Phi(\alpha)\) establishes (5). Inequality (6) follows from (4), and (5) with \(E_0 = E\).

It may seem surprising that the second member of (5) is not \(1/\rho\), for \(R_n(z)\) approaches zero and is merely of the form \(B_n/(z - \alpha_n)\) where \(\alpha_n \to \alpha\). Theorem 4 of reference 1 asserts that such a function has the same degree of convergence \((n \to \infty)\) on every closed point set not containing \(\alpha\). However, there is in our hypothesis no presumption that \(R_{n\mu}(z) - R_{n\sigma}(z)\) has no pole at infinity; such a pole of order \(n\) must be considered a possibility. Here a counterexample is useful. Let \(E\) be \(|z| \leq 1/\rho\ (<1)\), \(\Gamma_\rho\) be \(|z| = 1\), let \(f(z)\) satisfy the hypothesis of Theorem 2, and set \(u = v + 1\).
\[ R_{n+r,n}(z) = R_{n,r}(z) + \frac{z^n}{z - \alpha_n}, \]  
(11)

where \( R_{n,\nu}(z) \) is the function of Theorem 1, chosen to exist for every \( n \). We may choose the \( \alpha_n \) in many ways; for the moment choose \( \alpha_n \to \alpha \) in \( E_{\rho} - E \). If the equality sign holds in (2), it also holds in (4), (5), and (6), with \( \Phi(z) \equiv \rho |z| \); we have \( R_n(z) \equiv \alpha_n^n/(z - \alpha_n) \).

Theorem 2 can be improved in the case \( \mu = \nu + 1 \), where we suppose (4) with

\[ R_{n+r,\nu}(z) = R_{n,\nu}(z) + \frac{B_n}{z - \alpha_n}, \]  
(12)

where \( R_{n,\nu}(z) \) satisfies (2). Indeed (5) with \( \Phi(\alpha) \) replaced by unity follows at once [i.e., without (10)] for \( z \) on \( E \), hence follows [ref. 1, Theorem 4] for \( E_0 \) any closed set of the plane not containing \( \alpha \). Inequality (6) is here weaker than the inequality (2).

If (12) is modified so that precisely \( k \) poles of \( R_{n+r,\nu}(z) \) approach \( \alpha \), \( k \leq \mu - \nu \), the same method shows that (5) with \( \Phi(\alpha) \) replaced by unity is valid for \( z \) on \( E \), hence follows as before for \( E_0 \) any closed set of the plane not containing \( \alpha \). Moreover, precisely \( k \) zeros of \( R_{n+r,\nu}(z) \) also approach \( \alpha \); compare Lemma 1 of reference 1.

The example (11) is also of interest if the \( \alpha_n \) are everywhere dense in \( E_{\rho} - E \) or on \( E_\sigma \), \( \sigma \leq \rho \); compare here [ref. 2, Theorem 5].

The proof of Theorem 2 with minor changes yields:

**Corollary 1.** Theorem 2 remains valid if the hypothesis is modified so that poles of \( R_{n,\nu}(z) \) of total multiplicity \( \lambda \) approach a point \( \alpha \) in \( E_{\rho} - E \) which is either a point of analyticity of \( f(z) \) or a pole of \( f(z) \), necessarily of multiplicity not greater than \( \lambda \).

Here \( R_n(z) \) denotes the sum of the principal parts of the totality of poles of \( R_{n,\nu}(z) \) approaching \( \alpha \) less the principal part of the pole (if any) of \( f(z) \) at \( \alpha \).

It may be noted that any subsequence of the \( R_{n,\nu}(z) \) whose poles have a limit point in \( E_{\rho} - E \) admits a subsequence satisfying the conditions of Theorem 2 or of Corollary 1. Iteration of this remark yields a subsequence of the \( R_{n,\nu}(z) \) having no more than \( \mu \) limit points of poles in \( E_{\rho} - E \), including the \( \nu \) poles of \( f(z) \) in \( E_{\rho} - E \).

**Corollary 2.** Theorem 2 remains valid if the hypothesis is modified so that the limit points of some or all of the extraneous poles of \( R_{\nu,\nu}(z) \) in \( E_{\rho} - E \) (that is, poles in addition to the \( \nu \) poles approaching the \( \nu \) poles of \( f(z) \) in \( E_\rho \)) lie interior to a closed region \( E' \) in \( E_\rho \) which \( E' \) contains no other limit points of poles of \( R_{\nu,\nu}(z) \) except perhaps poles of \( f(z) \). Precise analogues of (5) and (6) are valid, where \( \Phi(\alpha) = [\max \Phi(z), z \in E'] \), and \( R_n(z) \) is the sum of the principal parts of the totality of poles of \( R_{\nu,\nu}(z) \) approaching points of \( E' \; \text{less} ;\) the principal parts of the poles of \( f(z) \) in \( E' \).

Corollary 2 applies even if \( E' \) is a closed multiply connected region, and contains points of \( E \) in a lacuna. Corollary 2 follows at once, except in the case (included below) that \( E' \) is a subset of \( E \).

**Corollary 3.** Let the hypothesis of Theorem 2 be modified so that some or all of the poles of \( R_{\nu,\nu}(z) \) in \( E_{\rho} \) approach \( E \), and suppose \( \beta \) in \( E_{\rho} - E \) can be chosen so that no poles of \( f(z) \) lie interior to \( \Gamma_{\Phi(\beta)} \), and no limit points of poles of \( R_{\nu,\nu}(z) \) lie there except on \( E \). Let \( R_n(z) \) denote the sum of the principal parts of the poles of \( R_{\nu,\nu}(z) \) interior to \( \Gamma_{\Phi(\beta)} \). Then (5) and (6) are valid with \( \Phi(\alpha) \) replaced by unity.
The proof of Theorem 2 is essentially sufficient to establish Corollary 3. Choose \( \gamma \) as \( \Gamma_\gamma \), \( 1 < \tau < \Phi(\beta) \). We use (10) for \( z \) on \( \Gamma_\Phi(\beta) \), and deduce
\[
\lim_{n \to \infty} \sup_n [\max |R_{n,z}(z)|, \text{z on} \Gamma_\Phi(\beta)]^{1/n} \leq \tau/\rho,
\]
whence by the method of proof of (9)
\[
\lim_{n \to \infty} \sup_n [\max |f(z) - [R_{n,z}(z) - R_{n}(z)]|, \text{z on} \Gamma_\Phi(\beta)]^{1/n} \leq \Phi(\beta)/\rho.
\]
The latter inequality yields \( \Phi(\beta) \to 1 \), the assertion regarding (6), and the assertion regarding (5) follows by (4).

**Corollary 4.** In Theorem 2 let the condition on the poles of the \( R_{n,z}(z) \) be deleted. Then on any closed set \( E_0 \) in \( E_z \) containing no limit points of poles of the \( R_{n,z}(z) \) and no pole of \( f(z) \) we have
\[
\lim_{n \to \infty} \sup_n [\max |f(z) - R_{n,z}(z)|, \text{z on} E_0]^{1/n} \leq \Phi(\alpha)/\rho,
\]
where \( \Phi(\alpha) = [\max \Phi(z), \text{z on} E_0] \).
The proof of (13) follows precisely the proof of (9).
The corollaries are clearly related to

**Theorem 3.** In Theorem 2 let the \( R_{n,z}(z) \) be defined for every \( n \), and modify the hypothesis so that all the limit points of the poles of the \( R_{n,z}(z) \) in \( E_\rho + \Gamma_\rho \) lie on the closed subset \( E' \) of \( E_\rho \). Then the equality sign holds in (4).

Suppose the second member of (4) is \( 1/\rho_1(<1/\rho) \); we shall reach a contradiction. We have
\[
\lim_{n \to \infty} \sup_n \|[R_{n+1,z}(z) - R_{n,z}(z)]^{1/n} \leq 1/\rho_1,
\]
and by a slight modification of Lemma 3 of reference 1,
\[
\lim_{n \to \infty} \sup_n [\max |R_{n+1,z}(z) - R_{n,z}(z)|, \text{z on} \Gamma_z] \leq \sigma/\rho_1,
\]
where \( \Gamma_z, \sigma < \rho < \rho_1 \), contains \( E' \) in its interior. Further use of essentially that lemma, with (14) as hypothesis, gives
\[
\lim_{n \to \infty} \sup_n [\max |R_{n+1,z}(z) - R_{n,z}(z)|, \text{z on} \Gamma_z]^{1/n} \leq \tau/\rho_1,
\]
by virtue of the equation \( [\Gamma_z]_{\rho_1} = \Gamma_z \); here we choose \( \rho < \tau < \rho_1 \), and also choose \( \tau \) so that no limit points of poles of the \( R_{n,z}(z) \) lie in the closed annular regions between and bounded by \( \Gamma_\rho \) and \( \Gamma_z \). Consequently the sequence \( R_{n,z}(z) \) converges uniformly on both \( \Gamma_\rho \) and \( \Gamma_z \), has no limit point of poles in the set of annular regions bounded by them, and converges uniformly to some analytic function \( F(z) \) in those annular regions. Corollary 4 implies \( F(z) \equiv f(z) \) in the parts of those annular regions in \( E_\rho \), hence also throughout those regions, contrary to the definition of \( \rho \) in Theorem 2.

Under the conditions of Theorems 1 and 2, the class \( R_{n,z}(z) \) includes the class \( R_{n}(z) \), so for the respective functions of best approximation to \( f(z) \) on \( E \) we have
\[
\|f(z) - R_{n,z}(z)\| \leq \|f(z) - R_{n}(z)\|.
\]
Nevertheless, for the functions of best approximation the first member of (2) is
1/\rho$, and under the conditions of Theorem 3 the first member of (4) is also $1/\rho$.

Theorem 2 and later results including the corollaries have application to the dual problem, approximation by functions of type $R_{\infty}(z)$ considered in reference 3.

The methods of the present note for the study of convergence and degree of convergence on subsets of $E_\rho$ apply when the number of free poles of the approximating rational functions is finite, and by dualization apply when the number of their free zeros is finite; they do not apply when the number of both free poles and zeros is infinite.

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RANDOM POLYGONS DETERMINED BY RANDOM LINES IN A PLANE*

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Introduction.—Consider a system of straight lines in different directions, distributed at random homogeneously throughout a plane (see Fig. 1). This paper presents the main results of a study concerned primarily with the statistical properties of the aggregates of polygons into which such systems divide the plane. This problem was first considered by Goudemit, who was able to derive a number of properties. The present study utilizes more powerful and more general methods, and consequently yields many additional results. In order to give a brief and simplified description of the theory comprehensible to nonspecialists interested in applications, a number of mathematical points must needs be glossed over or even ignored; the following treatment should therefore only be regarded as heuristic. Parentheses will often signify heuristic ideas. Only a knowledge of elementary probability theory is presupposed. The precise specifications of the system are followed by the sequence of main results. The second and concluding part of this paper, which will appear in the November issue of these PROCEEDINGS, contains a short account of methods and of an application.

The Line System $£$.—By way of notation, the equation of any line in the $(x,y)$ plane may be written as

$$p = x \cos \theta + y \sin \theta \quad (-\infty < p < \infty, 0 \leq \theta < \pi),$$

where $p$ is the signed (i.e., positive or negative) length of the perpendicular, or its distance, from the origin $O$ and $\theta$ is its orientation—the angle this perpendicular makes with $Ox$. There is thus a one-to-one correspondence between lines in the $(x,y)$ plane and points in the strip $0 \leq \theta < \pi$ of the $(p,\theta)$ plane.

The distances $\ldots \leq p_{-2} \leq p_{-1} \leq p_0 \leq p_1 \leq p_2 \leq \ldots$ of the lines from $O$ constitute the coordinates of the events of a Poisson (or purely random) process on a one-dimen-