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*AN ELEMENTARY THEORY OF THE CATEGORY OF SETS**

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We adjoin eight first-order axioms to the usual first-order theory of an abstract Eilenberg-Mac Lane category[1] to obtain an elementary theory with the following properties: (*a*) There is essentially only one category which satisfies these eight axioms together with the additional (nonelementary) axiom of completeness, namely, the category $\mathcal{S}$ of sets and mappings. Thus our theory distinguishes $\mathcal{S}$ structurally from other complete categories, such as those of topological spaces, groups, rings, partially ordered sets, etc. (*b*) The theory provides a foundation for number theory, analysis, and much of algebra and topology even though no relation $\in$ with the traditional properties can be defined. Thus we seem to have partially demonstrated that even in foundations, not Substance but invariant Form is the carrier of the relevant mathematical information.

As in the general theory of categories, our undefined terms are *mapping*, *domain*, *codomain*, and *composition*, the first being simply a name for the elements of the universe of discourse. Each mapping has a unique domain and a unique codomain, these being also mappings of the special sort called *objects*. We write $A \xrightarrow{f} B$ iff A is the domain of f and B is the codomain of f. Given any pair f, g of mappings such that the codomain of f is the domain of g, there is a unique mapping denoted by fg and called the *composition* of f with g; the domain of fg is the domain of f and its codomain is that of g. Composition is associative and objects behave as neutral elements with respect to it. A mapping is an *isomorphism* iff it has a two-sided inverse.

AXIOM 1 (Finite Roots). *There is a terminal object* 1 *and an initial object* 0. *Every pair* A_1, A_2 *of objects has a product with projections* $A_1 \times A_2 \xrightarrow{p_k} A_k$ *and a sum with injections* $A_k \xrightarrow{i_k} A_1 + A_2$ *satisfying the usual universal mapping properties. Every pair* $A \overset{f}{\underset{g}{\rightrightarrows}} B$ of *mappings has an equalizer* $K \to A$ *and a coequalizer* $B \to Q$ *satisfying the usual universal mapping properties.*

Definition 1: x is an *element* of A iff $1 \xrightarrow{x} A$. (We use the usual notation $x \in A$ for this as well as for the more general notion of membership defined below, even though these relations have few formal properties in common with the traditional one of set theory.)

AXIOM 2 (Exponentiation). *For any pair* A, B *of objects there is an object* B^A *and an "evaluation" mapping* $A \times B^A \xrightarrow{e} B$ *with the property that given any object* X *and any mapping* $A \times X \xrightarrow{f} B$, *there is a unique* $X \xrightarrow{h} B^A$ *such that* $(A \times h)e = f$.

Here "$A \times h$" refers to the natural functorial extension of the product operation from objects to mappings; such an extension exists as well for the sum and exponentiation operations. Of course, all these operations are well-defined up to a unique natural isomorphism. It follows easily that the elements of a product are ordered pairs, that the values of an equalizer are precisely the elements at which the two maps agree, and that the elements of B^A are in canonical one-to-one correspondence with the mappings $A \to B$. The nature of the elements of a sum or of a coequalizer are not so easy to discover, but will be clarified below. The usual laws of exponents hold (up to a canonical isomorphism) and the existence of exponentiation implies that products are distributive over sums.

AXIOM 3. *There is an object* N *together with mappings* $1 \xrightarrow{z} N \xrightarrow{s} N$ *such that given any object* X *together with mappings* $1 \xrightarrow{x_0} X \xrightarrow{t} X$, *there is a unique mapping* $N \xrightarrow{x} X$ *such that* $x_0 = zx$ *and* $xt = sx$.

Clearly there is, up to canonical isomorphism, only one such system $1 \xrightarrow{z} N \xrightarrow{s} N$. Mappings $N \to X$ are called *sequences*, and the sequence x of the axiom is said to be defined by *simple recursion* from the recursion data x_0, t. Of course, for mathematics we need more complicated sorts of recursions, so it is fortunate that we can prove

THEOREM 1 (Primitive Recursion). *Given mappings*

$$A \xrightarrow{f_0} B$$

$$N \times A \times B \xrightarrow{u} B$$

there is a mapping

$$N \times A \xrightarrow{f} B$$

such that for any $a \in A$, $n \in N$ *one has*

$$<0, a> f = af_0$$

$$<ns, a> f = <n, a, <n, a>f>u.$$

The proof is a straightforward application of Axiom 2, Axiom 3, and the existence of products from Axiom 1; the following axiom implies that the f of the theorem is unique.

AXIOM 4 (1 is a generator). *If mappings* $A \underset{g}{\overset{f}{\rightrightarrows}} B$ *are different, then there is an element* $x \in A$ *such that* $xf \neq xg$.

AXIOM 5 (Axiom of Choice). *If the domain of* f *has elements, then there is* g *such that* $fgf = f$.

The independence of the axiom of choice is easy to see in our theory, for the category $\mathcal{O}$ of partially ordered sets and order preserving maps satisfies the other seven axioms but not Axiom 5. The same example shows that the use of the axiom of choice is essential even in the proof of such basic propositions as Proposition 2 below.

PROPOSITION 1. $1 + 1$ *is a cogenerator.*

Definition 2: a is a *subset* of A iff a is a monomorphism with codomain A.

Definition 3 (Membership): $x \in a$ iff for some (unique) A, x is an element of A, a is a subset of A, and there exists $\bar{x}$ such that $x = \bar{x}a$.

Definition 4 (Inclusion): $a \subseteq b$ iff a and b are both subsets of the same set and for some h, $a = hb$.

PROPOSITION 2. *If* a, b *are subsets of the same set, then*

$$a \subseteq b \Longleftrightarrow \forall\, x\ [x \in a \Longrightarrow x \in b].$$

AXIOM 6. *If* A *is not an initial object, then* A *has elements.*

AXIOM 7. *An element of a sum is a member of one of the injections.*

AXIOM 8. *There exists an object with more than one element.*

This completes the list of axioms. Axiom 8 is clearly independent, since Axioms 1–7 are satisfied in the category with only one mapping.

PROPOSITION 3. $x \notin 0$.

PROPOSITION 4. $1 + 1 = 2$. *That is, the two injections* $1 \xrightarrow{i_k} 1 + 1$ *are different and* $1 + 1$ *has no other elements.*

PROPOSITION 5. *In* $A + B$ *the subsets* i_A *and* i_B *have no members in common.*

THEOREM 2. *All of Peano's Postulates hold for* N.

THEOREM 3 (Regular Coimage $\cong$ Regular Image). *For any mapping* f, *let* q *denote the coequalizer of* kp_0 *with* kp_1, *where* k *is the equalizer of* p_0f *with* p_1f, *and let* q^* *denote the equalizer of* i_0k^* *with* i_1k^*, *where* k^* *is the coequalizer of* fi_0 *with* fi_1 *(here the* p*'s and* i*'s are projections and injections)*

$$\begin{tikzcd} K \arrow[r, "k"] & A \times A \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & B + B \arrow[r, "k^*"] & K^* \\ & & I^* \arrow[r, "h"] & I \arrow[u, "q^*"'] & & \end{tikzcd}$$

Then the canonical h in the above diagram is an isomorphism. Hence we may assume that $I^* = h = I$ *and refer to the equation* $f = qq^*$ *as the factorization of f through its image.*

Definition 5: ϕ is the *characteristic function* of a, where a is a subset of X, iff $X \xrightarrow{\phi} 2$ and for every $x \in X, x\phi = i_1$ iff $x \in a$.

Here we assume that a standard labeling of the two injections $1 \to 2$ will be chosen during any discussion involving characteristic functions. Using equalizers, it is clear that every $X \to 2$ is the characteristic function of some subset. Conversely, Theorem 5 states that every subset has a characteristic function. First we need to know that (in any model for our theory) certain infinite unions exist; namely, a union exists for any family of subsets of a given set which has the properties that each subset in the family has a characteristic function and that there exists a single mapping α (in the model) which parameterizes the family.

THEOREM 4. *To every mapping* $I \xrightarrow{\alpha} 2^X$ *there is a subset* $\bigcup \alpha \xrightarrow{a} X$ *such that for any* $x \in X$, *one has* $x \in a$ *iff there is* $j \in I$ *such that* $x \in \alpha_j$, *where* α_j *is the subset of X whose characteristic function corresponds (à la Axiom 2) to the element* $j\alpha$ *of* 2^X.

Remark: Given $I \xrightarrow{\alpha} 2^X$, it is easy to construct the subset $\Sigma\alpha \to X \times I$ (whose members are just the pairs $<x, j>$ for which $x \in \alpha_j$) and the subset $\Pi\alpha \to (\Sigma\alpha)^I$ consisting of all "choice functions" $I \to \Sigma\alpha$. If the values of α are nonempty, then there is at least one such "choice function," which upon composing with the projection gives a mapping $I \xrightarrow{f} X$ such that $jf \in \alpha_j$ for all $j \in I$.

THEOREM 5. *Every subset* $A \xrightarrow{a} X$ *has a complement* $A' \xrightarrow{a'} X$ *in the sense that the induced map* $A + A' \to X$ *is an isomorphism. Hence every subset has a characteristic function.*

The idea of the proof of Theorem 5 is to define a' as the union of all those subsets of X which do have characteristic functions and which do not intersect a. To complete the proof we use a lemma which states that there is at least one such subset containing as a member any element x of X for which $x \notin a$.

The fact about coequalizers in $\mathcal{S}$ which is stated in the next theorem was previously shown to be characteristic of "algebraic" categories.[2] For the proof we need in our present theory lemmas guaranteeing the existence of the *singleton* mapping $A \to 2^A$ and of the covariant *direct-image* power-set functor.

In the terminology of the previous paper[2] the statement of Theorem 6 is "every precongruence is a congruence." Before stating the theorem we give for the relevant concepts definitions which are meaningful in an arbitrary category, so in particular in any model for our theory.

Definition 6: A pair $R \underset{f_1}{\overset{f_0}{\rightrightarrows}} A$ of mappings is

$$\textit{reflexive} \text{ iff } \exists d[A \xrightarrow{d} R \;\&\; df_0 = A = df_1]$$

$$symmetric \text{ iff } \exists t[R \xrightarrow{t} R \;\&\; tf_0 = f_1 \;\&\; tf_1 = f_0]$$

$$transitive \text{ iff } \forall h_0 \forall h_1 [X \underset{h_1}{\overset{h_0}{\rightrightarrows}} R \;\&\; h_0 f_1 = h_1 f_0 \Rightarrow \exists u[uf_0 = h_0 f_0 \;\&\; uf_1 = h_1 f_1]]$$

a *relation* iff $\forall y \forall y' [yf_0 = y'f_0 \;\&\; yf_1 = y'f_1 \Rightarrow y = y']$. Thus in the present theory $R \underset{f_1}{\overset{f_0}{\rightrightarrows}} A$ is a relation iff the induced mapping $R \xrightarrow[<f_0,f_1>]{} A \times A$ is a subset of $A \times A$.

THEOREM 6. *If $R \underset{f_1}{\overset{f_0}{\rightrightarrows}} A$ is a reflexive, symmetric, and transitive relation, then $f = <f_0,f_1>$ is the equalizer of p_0q with p_1q, where is q the coequalizer of f_0 with f_1 and p_0,p_1 are the projections.*

The method of proof is to construct the partition map $A \rightarrow 2^A$.

Remark: From Theorem 6 it quickly follows that the coequalizer of any pair of maps is also the coequalizer of the RST hull of the given pair. The latter can be constructed in a more element-wise fashion by using N.

Finally, we derive the promised completeness theorem from a more general metatheorem about models of our theory by making use of some well-known facts about adjoint functors.[3, 4]

METATHEOREM. *Let $\mathbf{C} \xrightarrow{T} \mathbf{C}'$ be a functor such that both* $\mathbf{C}$ and $\mathbf{C}'$ *satisfy the eight axioms and*

(1) *both $\mathbf{C}$ and $\mathbf{C}'$ have the property that for each object A, the lattice of subobjects of A is complete,*

(2) *T has an adjoint $\check{T}$,*

(3) *for each A in $\mathbf{C}$, the induced mapping $(1,A)_{\mathbf{C}} \rightarrow (1T,AT)_{\mathbf{C}'}$ is surjective, where $(B,A)_{\mathbf{C}}$ means the set of maps $B \rightarrow A$ in $\mathbf{C}$. Then T is an equivalence of categories, i.e., $T\check{T}$ and $\check{T}T$ are naturally equivalent to the respective identity functors.*

By setting $\mathbf{C}' = \mathcal{S}$, $T = (1,-)_{\mathbf{C}}$ and $S\check{T} = \underset{S}{\Sigma} 1$, we then have the

COROLLARY. *If $\mathbf{C}$ is a complete category satisfying the eight axioms, then $\mathbf{C}$ is equivalent to $\mathcal{S}$.*

Here completeness has the usual meaning of category theory, namely, that infinite products and sums over any indexing set exist. The extent to which completeness can fail is indicated by the fact that the set of all mappings between sets of rank less than $\omega + \omega$ is a model for our theory. Actually no first-order theory can guarantee even that the operation $A \rightsquigarrow \underset{N}{\Sigma} A$ exists, for it is easy to see that any nontrivial category in which this operation exists must be nondenumerable (of course this functor is naturally equivalent to $A \rightsquigarrow N \times A$ when it does exist).

It is easy to add to our theory axioms which guarantee the existence of cardinals much larger than $\aleph_\omega$, although these are almost never needed, say for analysis. (But such axioms would at least exclude the model of the previous paragraph.) However, it is the author's feeling that when one wishes to go substantially beyond what can be done in the theory presented here, a more satisfactory foundation will involve a theory of the category of categories.

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THE CONSTRUCTION OF FORMAL COHOMOLOGY SHEAVES

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Introduction.—Let R be a complete discrete valuation ring of characteristic zero, $\mathfrak{m}$ the valuation ideal of R, $k = R/\mathfrak{m}$ the residue field, and Q_R the fraction field of R. The principal case of interest arises if R is the completion of a ring of algebraic integers with respect to one of its prime ideals; but the equal characteristic case or even the particular case where R is a field and $\mathfrak{m}$ is the zero ideal is not entirely without interest.

Let V be a nonsingular variety defined over k. We shall attach to V certain sheaves $\mathbf{H}^l(V)$ of Q_R-modules—the "formal cohomology sheaves" of V with respect to the valuation ring R. These sheaves are defined for all integers $l \geq 0$ but they are zero for $l > \dim V$; their base space is V (with the Zariski topology). Furthermore, if V and W are nonsingular varieties defined over k and φ: $V \rightarrow W$ is a morphism, we shall construct a natural homomorphism of sheaves of Q_R-modules

$$\varphi^l{}_*: \mathbf{H}^l(W) \rightarrow \varphi_*\mathbf{H}^l(V),$$

where $\varphi_*\mathbf{H}^l(V)$ is the sheaf of Q_R-modules on W that is the direct image of $\mathbf{H}^l(V)$ with respect to φ: $V \rightarrow W$; put another way, for each open set U in W there is a natural homomorphism from the module of sections of $\mathbf{H}^l(W)$ over U into the module of sections of $\mathbf{H}^l(V)$ over $\varphi^{-1}(U)$, and these homomorphisms commute with restrictions. Finally, the direct sum sheaf $\mathbf{H}^*(V) = \bigoplus_{l=0}^{\infty} \mathbf{H}^l(V)$ has the structure of a sheaf of graded Q_R-algebras with anticommutative multiplication; the component of degree l is $\mathbf{H}^l(V)$ and $\varphi_* = \bigoplus \varphi_*{}^l$ preserves this multiplication. Thus $(V,\mathbf{H}^*(V))$ is a ringed space of graded anticommutative Q_R-algebras, and $(V,O_V) \rightarrow (V,\mathbf{H}^*(V))$ is a functor from the category of nonsingular varieties defined over k to the category of ringed spaces of graded anticommutative Q_R-algebras.

1. A "special affine variety" is a variety obtained from an irreducible affine hypersurface $f(X_1,\ldots, X_n,Y) = 0$ by deleting the section $\partial f/\partial Y = 0$. The coordinate ring A of a special affine variety defined over k has the form $k[X_1,\ldots, X_n,Y,Z]/I$ where I is the ideal generated by $f(X_1,\ldots, X_n,Y)$ and $Z(\partial f/\partial Y) - 1$ and f is irreducible. Such an A is called a "special affine k-algebra." The special affine open sets on a given nonsingular variety form a neighborhood base for the Zariski topology. For this reason it suffices to use them to construct the formal cohomology sheaves, but for further developments of the theory (e.g., formal cohomology of product varieties) we find it necessary to extend the class of special affine varieties to include principal open subsets on products of special affine