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*NONEXPANSIVE NONLINEAR OPERATORS IN A BANACH SPACE**

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Let X be a Banach space, C a closed bounded convex subset of X. If U is a mapping (generally nonlinear) of C into X, U is said to be *nonexpansive* if for each pair of elements x and y of C, we have

$$||Ux - Uy|| \leq ||x - y||.$$

In the present note, we give the proof of the following two theorems:

THEOREM 1. *Let X be a uniformly convex Banach space, U a nonexpansive mapping of the bounded closed convex subset C of X into C. Then U has a fixed point in C.*

THEOREM 2. *Let X be a uniformly convex Banach space, $\{U_\lambda\}$ a commuting family of nonexpansive mappings of a given bounded closed convex subset C of X into C. Then the family of mappings $\{U_\lambda\}$ has a common fixed point in C.*

In two recent notes[1, 2] in these PROCEEDINGS, the writer proved Theorem 1 for the case when X is a Hilbert space using the concepts of the theory of monotone operators in Hilbert space as developed by the writer[3] and G. J. Minty.[4] The writer extended this result to the class of spaces having weakly continuous duality mappings[5] and T. Kato gave an independent proof for the spaces l^p, $1 < p < \infty$ (for which weakly continuous duality mappings exist). The result we give here is much more general, since it is valid for the L^p-spaces, $1 < p < +\infty$, for which the duality mappings are not weakly continuous ($p \neq 2$). Theorem 2 is a nonlinear extension of the Markov-Kakutani theorem[6] for linear mappings and an extension of the result of De Marr[7] for C compact.

If C is compact or U is compact, Theorem 1 is a consequence of the Schauder fixed-point theorem, while if U is weakly continuous, it is a special case of the Tychonoff fixed-point theorem since in a reflexive space, every bounded closed convex set C is weakly compact. If U is a strict contraction (i.e., if there exists a constant $k < 1$ such that $\|Ux - Uy\| \leq k\|x - y\|$), the result follows from the classical contraction principle of Picard.[8] The force of Theorems 1 and 2 lies in the fact that they are valid without such additional hypotheses and are therefore applicable in more general situations. We have given an application in our preceding note[1] to the proof of the existence of periodic solutions of a very general class of nonlinear equations of evolution in infinite dimensional spaces.

Let us remark before proceeding to the proofs of Theorems 1 and 2 that the following simple example (which the writer owes to Richard Beals) shows that these results cannot be extended to the general class of Banach spaces: Let $X = c_0$, the space of sequences converging to zero, C the unit ball in the maximum norm, e_1 the unit vector with first component 1 and others zero, $s(x) = (0, x_1, x_2, \ldots)$. Then the mapping $U(x) = e_1 + s(x)$ maps C into itself, is nonexpansive, and has no fixed point in C.

Proof of Theorem 1: Let Φ be the family of nonempty closed convex subsets of C which are invariant under U, i.e., $U(C_1) \subset C_1$ for $C_1 \epsilon \Phi$. Φ is nonempty since C is an element of Φ. If we order Φ by defining $C_1 \leq C_2$ if $C_1 \subset C_2$, Φ becomes a partially ordered set which is inductive since the intersection of the elements C_α of a linearly ordered subfamily of Φ is also a closed convex subset C' of C which is invariant under U and which is nonempty since all the sets C_α are weakly compact subsets of the reflexive Banach space X. Hence Φ has a minimal element C_0.

We remark that C_0 is the convex closure of $T(C_0)$, since if C_1 is this convex closure, C_1 is a closed convex subset of C_0 which is nonempty and invariant under U and by the minimality of C_0 in Φ, $C_0 = C_1$.

To complete the proof, it suffices to show that C_0 has exactly one element. Since C_0 is nonempty, it suffices to show that C_0 does not have two distinct points in it. Suppose it does. Let d_0 be the diameter of C_0, and choose two points x_1 and x_2 in C_0 such that $\|x_1 - x_2\| \geq d_0/2$. Let x be the midpoint of the segment joining x_1 to x_2, so that $x \epsilon C_0$. For any element y of C_0, we remark that $x - y$ is the midpoint of the segment joining $(x_1 - y)$ to $(x_2 - y)$, and $\|x_1 - y\|$ and $\|x_2 - y\|$ are both $\leq d_0$. By the uniform convexity of the space X, there exists a constant $q > 0$ such that

$$\|x - y\| \leq (1 - q)d_0 < d_0,$$

(since $\|(x_1 - y) - (x_2 - y)\| = \|x_1 - x_2\| \geq d_0/2$). Let $d_1 = (1 - q)d_0 < d_0$, and let

$$C_2 = \bigcap_{y \epsilon C_0} \{u|u \epsilon C_0, \|u - y\| \leq d_1\}.$$

C_2 is a closed convex subset of C_0 since it is the intersection of closed convex sets, and is nonempty since x lies in C_2. C_2 is a proper subset of C_0 since d_1 is less than the diameter of C_0. Finally, C_2 is invariant under U. Indeed, suppose $u \epsilon C_2$, $y \epsilon C_0$. For any $\epsilon > 0$, we can find a convex linear combination of $U(z_j)$, $z_j \epsilon C_0$, such that

$$\|y - \Sigma_{j=1}^{r} \lambda_j U(z_j)\| < \epsilon, \ (0 \leq \lambda_j \leq 1, \ \Sigma_{j=1}^{r} \lambda_j = 1).$$

Thus,

$$\|Uu - y\| \leq \|Uu - \Sigma_j \lambda_j U z_j\| + \epsilon$$

$$\leq \Sigma_j \lambda_j \| Uu - Uz_j \| + \epsilon,$$

while

$$\| Uu - Uz_j \| \leq \| u - z_j \| \leq d_1,$$

since U is nonexpansive and u lies in C_2. Hence,

$$\| Uu - y \| \leq \Sigma_j \lambda_j d_1 + \epsilon = d_1 + \epsilon$$

for each $\epsilon > 0$, which implies that $\| Uu - y \| \leq d_1$ for all y in C_0, i.e., Uu lies in C_2.

However, C_0 is a minimal element of Φ, so that we have reached a contradiction from the assumption that C_0 has at least two points. Hence, $C_0 = \{x_0\}$ and $Ux_0 = x_0$. q.e.d.

Proof of Theorem 2: If U is a nonexpansive mapping in a strictly convex space, the fixed-point set of U is a closed convex set (since if y lies on the segment joining two fixed points y_0 and y_1 of U, we have

$$\| Uy - y_0 \| + \| Uy - y_1 \| \leq \| y - y_0 \| + \| y - y_1 \| = \| y_1 - y_0 \|).$$

Hence, the fixed-point set F_λ of each U_λ is closed and convex, and nonempty by Theorem 1. If $u \epsilon F_\lambda$, then for any α,

$$U_\lambda(U_\alpha u) = U_\alpha(U_\lambda u) = U_\alpha u,$$

i.e., $U_\alpha u$ lies in F_λ and each U_α maps F_λ into itself. If we are given a finite sequence $\lambda_1, \ldots, \lambda_m$, considering U_{λ_m} as a nonexpansive mapping of $F_{\lambda_1} \cap F_{\lambda_2} \cap \ldots \cap F_{\lambda_{m-1}}$ into itself (where the latter is assumed nonempty), it follows from Theorem 1 that

$$\bigcap_{s=1}^{m} F_{\lambda_s} \neq \phi.$$

However, the F_λ, being closed convex bounded subsets of the reflexive space X, are weakly compact. Having the finite intersection property, the family $\{F_\lambda\}$ has a nonempty intersection, but this intersection consists precisely of the common fixed points of the mappings U_λ.[9] q.e.d.

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[8] Some extensions of the Picard contraction principle to nonexpansive mappings of metric and

Banach spaces have been given by Edelstein but under hypotheses on the existence of strong limit points for the sequence $\{U^n x\}$ which are difficult to verify. Cf. Edelstein, M., "On fixed and periodic points under contractive mappings," *J. London Math. Soc.*, **37**, 74–79 (1962), "On nonexpansive mappings of Banach spaces," *Proc. Cambridge Philos. Soc.*, **60**, 439–447 (1964).

[9] *Added in proof:* After the present note was transmitted for publication, the writer received copies of the manuscripts of two unpublished papers, by W. A. Kirk, and by W. A. Kirk and R. Belluce, respectively, obtaining similar results. In Kirk's paper, it is remarked that the general line of argument is due to Brodskii and Milman [*Dokl.* (1948)], who proved Theorem 1 for isometries. It should also be remarked that in another recent note by the writer [Browder, F. E., "Mapping theorems for noncompact nonlinear operators in Banach spaces," these PROCEEDINGS, **54**, 337–342 (1965)], results are obtained which in the case of Hilbert spaces imply that the conclusion of Theorem 1 is also valid for *semicontractive* mappings, i.e., mappings U of C into C where $Ux = S(x,x)$ and S is a mapping of $C \times C$ into C which is nonexpansive in its first variable and completely continuous in the second variable. As we show elsewhere, this extension remains valid for Banach spaces having weakly continuous duality mappings. It is not clear at the time of writing whether this extension is also valid for the general class of uniformly convex spaces.

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*ISOPIESTIC TECHNIQUE FOR MEASURING LEAF WATER POTENTIALS WITH A THERMOCOUPLE PSYCHROMETER**

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Measurements of the water potential of plant tissue are made with thermocouple psychrometers by enclosing the tissue and thermocouple in a small container kept at a constant temperature and determining the degree of cooling of the thermocouple as water evaporates from it and is absorbed by the tissue. It is assumed that the rate of vapor transfer is proportional to the difference in potential between the thermocouple and plant material. Rawlins[1] suggested that leaf water potentials obtained with Richards and Ogata psychrometers[2] are too high, indicating too low a water stress, because of the leaf resistance to the diffusion of water vapor. In contrast, Barrs[3] concluded that Richards psychrometer values, when corrected for heat of respiration,[4] are the same as those obtained with Spanner[5] psychrometers and are therefore free of leaf resistance error. The lack of agreement in these studies indicates a need for further examination of the leaf resistance error in psychrometer measurements.

The mathematics describing the diffusion process within a Richards psychrometer chamber (see below) indicates that there must be a leaf resistance error in the water potentials measured by this method. The close agreement in water potentials measured with the Richards and Spanner techniques suggests that the Spanner technique may also be in error. In this paper, we describe a new technique which is free of error attributable to leaf resistance, and this technique is used to evaluate the error in Richards and Spanner measurements.

The new technique involves the use of a modified Richards thermocouple to determine the rate of vapor movement between the thermocouple and the leaf when solutions of various potentials are present on the thermocouple. No vapor movement occurs when the solution on the thermocouple has a potential equal to that