This last method, use of (11) with a power (15) of (14), seems very convenient in the application of (13).

* Research sponsored (in part) by Air Force Office of Scientific Research.

[1] Elliott, H. M., "On approximation to functions satisfying a generalized continuity condition," *Trans. Am. Math. Soc.*, **71**, 1 (1951).

[2] Jackson, D., "On certain problems of approximation in the complex domain," *Bull. Am. Math. Soc.*, **36**, 851 (1930).

[3] Sewell, W. E., "Degree of approximation by polynomials in the complex domain," *Annals of Mathematics Studies* (Princeton, 1942), no 9.

[4] Walsh, J. L., "Interpolation and approximation," in *Coll. Pubs. Am. Math. Soc.* (1935), vol. 20.

[5] Walsh, J. L., and H. G. Russell, "Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions," *Trans. Am. Math. Soc.*, **92**, 355 (1959).

[6] Walsh, J. L., W. E. Sewell, and H. M. Elliott, "On the degree of polynomial approximation to harmonic and analytic functions," *Trans. Am. Math. Soc.*, **67**, 381 (1949).

[7] Walsh, J. L., and W. E. Sewell, *Trans. Am. Math. Soc.*, **49**, 229 (1941).

[8] Walsh, J. L., "The convergence of sequences of rational functions of best approximation, III," in preparation.

HEREDITARY RINGS

BY LANCE W. SMALL

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY

Communicated by S. S. Chern, November 29, 1965

Kaplansky[2] and the author[3] have given examples of right hereditary rings which were not left hereditary. In both instances the left global dimension was two. Indeed, in every known case of a ring with differing global dimensions, the difference was one. In this note we present an example of a right hereditary ring with left global dimension equal to three. This ring has the additional property of not being left semihereditary; thus, we sharpen an example of Chase.[1]

In the second section of this note we extend certain results of Chase to Noetherian hereditary rings. In particular, many theorems about Artinian hereditary rings, with minor modifications, are seen to be true for Noetherian hereditary rings.

1. *An Example in Hereditary Rings.*—Kaplansky constructed a ring A which has the following properties: (*i*) A is an algebra over a field F, (*ii*) A is von Neumann regular, (*iii*) the right ideals of A are countably generated, and (*iv*) there is a left ideal L such that the homological dimension (hd_A) of L is 1.

The left A-module $B = A/L$ may also be considered as a (right) vector space over F. Let T be the ring of all two-by-two "matrices" of the form:

$$\begin{pmatrix} a & b \\ 0 & f \end{pmatrix}, \qquad a \in A, \qquad b \in B, \qquad f \in F$$

where addition is componentwise and multiplication is given by:

$$\begin{pmatrix} a & b \\ 0 & f \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & f' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bf' \\ 0 & ff' \end{pmatrix},$$

T is an associative ring with unit.

THEOREM 1. *T is right hereditary and the left global dimension of T is three.*

Proof: Set $N = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| b \in B\right\}$. It is quickly checked that N is a left ideal of T. From Theorem 2.1 of reference 1, it follows that $hd_T N = 2$; thus, the left global dimension of T is at least three. By an inspection of the left ideals of T, it is not hard to see that the left global dimension of T is 3.

The right ideals of T have the form:

$$\left\{\begin{pmatrix} a & x \\ 0 & y \end{pmatrix} \middle| a \in I, \quad (y,x) \in S\right\}$$

where I is a right ideal of A, and S is a subspace of $F \oplus B$ such that $IB \subset S$. If K is a right ideal in which there is no element $k = \begin{pmatrix} a & x \\ 0 & y \end{pmatrix}$ with $a \neq 0$, then K is projective by the aforementioned result of Chase.

Suppose M is a right ideal and

$$M = \left\{\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix} \middle| a \in I, \quad c \in C\right\}$$

where I is a right ideal of A, and C is a subspace of B. Kaplansky has shown that there exists a sequence of idempotents $e_1, e_2, \ldots, e_n, \ldots$ in I such that $e_1 A \subset e_2 A \subset \ldots \subset e_n A \subset \ldots$ and $I = \bigcup_{n=1}^{\infty} e_n A$. Set $J_n = \begin{pmatrix} e_n & 0 \\ 0 & 0 \end{pmatrix} M$. It is easily seen that $J_1 \subset J_2 \subset \ldots \subset J_n \subset \ldots$ and each J_n is a direct summand of J_{n+1}. Again by Kaplansky's results, $M_1 = \bigcup_{n=1}^{\infty} J_n$ is projective. Let $M_2 = \left\{\begin{pmatrix} 0 & \bar{c} \\ 0 & 0 \end{pmatrix} \middle| \bar{c} \text{ in the orthogonal complement of } IC \text{ in } C\right\}$. M_2 is projective by remarks in the preceding paragraph. Finally, $M = M_1 \oplus M_2$ and M is projective.

The remaining case to check is the one in which nonzero entries are allowed in both the upper left corner and the lower right corner. Let D be such a right ideal, I the associated right ideal in A, and S the subspace in $F \oplus B$. One checks that $D = D_1 \oplus D_2$, where $D_1 = \left\{\begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix} \middle| a \in I, x \in IB\right\}$, and $D_2 = \left\{\begin{pmatrix} 0 & u \\ 0 & v \end{pmatrix} \middle| (v,u) \text{ in the orthogonal complement of } IB \text{ in } S\right\}$. By the previous cases both D_1 and D_2 are projective; hence, D is also projective.

Since B is a cyclic left A-module, N is a cyclic left ideal of T which is not projective; hence, T is not left semihereditary. By considering polynomial rings over T, we can find rings with left and right global dimensions $n + 2$ and n $(n \geq 1)$, respectively.

2. *Noetherian Hereditary Rings.*—In this section all "one-sided" conditions will be assumed to hold on the right; for example, Noetherian means right Noetherian. $N(R)$ will denote the maximal nilpotent ideal of the Noetherian ring R. Finally, $Q(R)$ will be the classical right quotient ring of R.

The major tools in our investigation will be the existence and properties of $Q(R)$ where R is a Noetherian hereditary ring.

THEOREM 2. *If R is a Noetherian hereditary ring, then $Q(R)$ exists and is an Artinian hereditary ring.*

Now that $Q(R)$ exists, we can exploit the results of Chase[1] on semiprimary rings.

Definition (Chase): Let R be a Noetherian ring. R is *triangular* if any complete set of mutually orthogonal primitive idempotents $e_1, e_2, \ldots, e_r$ can be indexed so that $e_iN(R)e_j = 0$ whenever $i \geq j$. A primitive idempotent e is one which cannot be written as a sum of two nontrivial orthogonal idempotents.

The following lemmas enable us to pull back results about $Q(R)$ to R.

LEMMA 1. *Let R be a Noetherian hereditary ring and e a nontrivial idempotent. eR is a minimal right annihilator and e is primitive if and only if $eQ(R)$ is an indecomposable right ideal of $Q(R)$.*

LEMMA 2. *If R is any ring for which $Q(R)$ exists and $R = I_1 \oplus \ldots \oplus I_n$ where the I_j are right ideals, then $Q(R) = I_1Q(R) \oplus \ldots \oplus I_nQ(R)$.*

We can now put the pieces together to show

THEOREM 3. *If R is a Noetherian hereditary ring, then R is triangular.*

From Theorem 3, we can find an idempotent e such that eRe is a ring with no nonzero nilpotent ideals, $eRe' = 0$ where $e' = 1 - e$, and $e'Re'$ is a ring for which $Q(e'Re')$ exists and is Artinian. R can be shown to be isomorphic to the ring consisting of all "matrices":

$$\begin{pmatrix} s & v \\ 0 & t \end{pmatrix}, \qquad s \in e'Re', \qquad t \in eRe, \qquad v \in e'Re$$

where addition and multiplication are defined as in section 1.

Using this last "structure" theorem and an induction on the number of isomorphism classes of simple modules of $Q(R)$, we can prove

THEOREM 4. *If R is a Noetherian hereditary ring, then $R/N(R)$ is again hereditary.*

We remark that all the theorems in this section remain true on replacing "hereditary" by "principal right ideals are projective." In order to obtain a complete "structure" theory for hereditary Noetherian rings, one would have to find some sort of theory for prime rings and that seems doubtful. However, if we take the prime rings as our basic building blocks, a little more can be said than was said following Theorem 3; but we shall not do it here.

[1] Chase, S. U., "A generalization of the ring of triangular matrices," *Nagoya Math. J.*, **18**, 13–25 (1961).

[2] Kaplansky, I., "On the dimension of modules and algebras, X," *Nagoya Math. J.*, **13**, 85–88 (1958).

[3] Small, L. W., "An example in Noetherian rings," these PROCEEDINGS, **54**, 1035–1036 (1965).