# ASYMPTOTIC DIOPHANTINE APPROXIMATIONS TO E*

By William W. Adams

UNIVERSITY OF CALIFORNIA, BERKELEY

*Communicated by S. Eilenberg, November 1, 1965*

1. *Statement of Results.*—Schmidt[5] proved that for almost all real numbers $\alpha$, the number of solutions in integers $p$, $q$ of the inequalities

$$|q\alpha - p| < 1/q \quad \text{and} \quad 1 \leqq q \leqq B$$

is asymptotic to a constant times log $B$. One might conjecture that the classical numbers (e.g., algebraic numbers, $e$, $\pi$) behave like almost all numbers. Machine computations[1] were carried out for some of these numbers, and they seemed to bear out such a conjecture. Also, Lang[3] has proved that the estimate is valid when $\alpha$ is a real quadratic irrationality.

In this paper, we shall obtain an asymptotic estimate for $e$, which shows that the conjecture for $e$ is false. The machine computations of reference 1 are misleading because the range in which they are carried out, even though going to $q \leqq 10^6$, is still too small to exhibit the proper asymptotic behavior.

The function $4^x\Gamma(x + 3/2)$ is strictly monotone-increasing. Let $G$ be its inverse function. An application of L'Hôpital's rule shows that

$$\lim_{x \to \infty} G(x) \Big/ \frac{\log x}{\log \log x} = 1.$$

Theorem 1. *Let $\lambda_1(B)$ be the number of solutions in integers $p$, $q$ of the inequalities*

$$|qe - p| < 1/q \quad \text{and} \quad 1 \leqq q \leqq B. \tag{1}$$

*Then*

$$\lambda_1(B) = \frac{(2G(B))^{3/2}}{3} + O(G(B)).$$

Theorem 2.—*Let $\lambda_2$ $(B)$ be the number of solutions in relatively prime integers $p$, $q$ of the inequalities* (1) *above. Then*

$$\lambda_2(B) = 3G(B) + O(1).$$

We note that if we let $\lambda_1{}^+(B)$ be the number of solutions in integers $p$, $q$ of the inequalities $0 < qe - p < 1/q$ and $1 \leqq q \leqq B$, then a trivial modification of the proof given below shows that $\lambda_1{}^+ = {}^1/_2\lambda_1$, and similarly for $\lambda_2{}^+$.

Theorems similar to these are true for any irrational number whose continued fraction expansion is similar to that of $e$. However, the notation is considerably more involved and so these extensions will be reserved for a later paper.

2. *Some Facts about Continued Fractions.*—The proofs of the theorems are based on the simple continued fraction for $e$. We recall some easily proved facts. See Khinchin[2] for details.

Let $\alpha > 0$ be any irrational number, and denote by $[a_0, a_1, a_2, \ldots]$ the simple continued fraction expansion of $\alpha$, where the $a_n$ are integers, positive for $n \geqq 1$. Denote the $n$th principal convergent to $\alpha$ by $p_n/q_n = [a_0, a_1, \ldots, a_n]$ and the $n$th intermediate convergents by $(p_n + rp_{n+1})/(q_n + rq_{n+1})$, where $1 \leqq r < a_{n+2}$. We shall refer to both types of convergents simply as convergents. We know that the even convergents form a strictly increasing sequence of rational numbers tending to $\alpha$, and the odd convergents form a strictly decreasing sequence tending to $\alpha$. Moreover, if $P/Q$ and $P'/Q'$ are two successive terms in either sequence, then $PQ' - P'Q = \pm 1$. We recall that $q_n = a_n q_{n-1} + q_{n-2}$, and hence, that $q_n/q_{n-1} = [a_n, a_{n-1}, \ldots, a_1]$. Further, we have always $|q_n\alpha - p_n| < 1/q_n$. Finally, let $\alpha_n = [a_n, a_{n+1}, \ldots]$. Then we have the easily derived formula

$$|(q_n + rq_{n+1})\alpha - (p_n + rp_{n+1})| = \frac{\alpha_{n+2} - r}{q_n + \alpha_{n+2}q_{n+1}}, \tag{2}$$

for integers $0 \leqq r < a_{n+2}$.

LEMMA. *If $|\alpha - p/q| < 1/q^2$ where $p$, $q$ are positive integers, then $p/q$ is a convergent of $\alpha$.*

*Proof:* We assume that $\alpha < p/q$, the other case being proved in a similar manner. If $p/q$ is not a convergent, then there exist two successive convergents $P/Q$ and $P'/Q'$ such that $\alpha < P/Q < p/q < P'/Q'$ and $P'Q - PQ' = 1$. Thus,

$$\frac{1}{q^2} > \frac{p}{q} - \alpha > \frac{p}{q} - \frac{P}{Q} \geqq \frac{1}{qQ},$$

and

$$\frac{1}{Q'q} \leqq \frac{P'}{Q'} - \frac{p}{q} < \frac{P'}{Q'} - \frac{P}{Q} = \frac{1}{Q'Q}.$$

These estimates are contradictory.

3. *Proof of the Theorem.*—We know[4] that $e = [2,1,2,1,1,4,1,1,6,1,1,8,\ldots]$ so that for $e$, we have $a_0 = 2$, $a_{3m} = a_{3m-2} = 1$, and $a_{3m-1} = 2m$ for all $m \geqq 1$. We set as above, $e_n = [a_n, a_{n+1}, \ldots]$.

PROPOSITION 1. *We have for all $n$, $m \geqq 1$,*

$$\lambda_1(q_{3m+1}) = {}^1/_3(2m)^{3/2} + O\,(m)$$

$$\lambda_2(q_n) = n + O(1).$$

*Proof:* We first show that, with a finite number of exceptions, the only reduced fractions $p/q$ satisfying $|e - p/q| < 1/q^2$ are the principal convergents. From this the formula for $\lambda_2$ is clear since the $n$th solution is simply $p_n$, $q_n$.

By the lemma we know that any solution must be a convergent, so suppose that $(p_n + rp_{n+1})/(q_n + rq_{n+1})$ with $0 \leqq r < a_{n+2}$ is a solution. We wish to show that $r = 0$. If $n = 3m - 2$ or $n = 3m - 1$, then $a_{n+2} = 1$ and so $r = 0$. Thus, we may restrict our attention to the case where $n = 3m$. To say that this convergent is a solution simply means by (2) that

$$\frac{e_{3m+2} - r}{q_{3m} + e_{3m+2}q_{3m+1}} < \frac{1}{q_{3m} + rq_{3m+1}}.$$

This condition is equivalent to $f(r) > 0$, where

$$f(r) = r^2 + \left(\frac{q_{3m}}{q_{3m+1}} - e_{3m+2}\right) r + \left(\frac{q_{3m}}{q_{3m+1}} + e_{3m+2} - e_{3m+2}\frac{q_{3m}}{q_{3m+1}}\right).$$

Thus, we must show that $f(1), f(2), \ldots, f(a_{3m+2} - 1) < 0$. Since $f$ is a quadratic polynomial in $r$ with leading coefficient 1, it suffices to show that $f(1) < 0$ and $f(a_{3m+2} - 1) < 0$. Well, $f(1) = 1 + (2 - e_{3m+2})q_{3m}/q_{3m+1}$, and $f(1) < 0$ follows from

$$e_{3m+2} > a_{3m+2} = 2(m+1), \qquad q_{3m+1}/q_{3m} = [1,1,2m,\ldots] < 2.$$

Furthermore,

$$f(a_{3m+2} - 1) = (e_{3m+2} - a_{3m+2})\,(2 - a_{3m+2} - q_{3m}/q_{3m+1}) + 1,$$

and $f(a_{3m+2} - 1) < 0$ follows similarly.

We must now determine which multiples of the convergents $p_n$, $q_n$ are also solutions of (1). Again by (2), with $r = 0$, this condition is equivalent to the condition

$$\frac{e_{n+2}}{q_n + e_{n+2}q_{n+1}} < \frac{1}{k^2 q_n} \quad \text{or} \quad k^2 < \frac{1}{e_{n+2}} + \frac{q_{n+1}}{q_n}.$$

If $n = 3m - 1$ or $n = 3m$, then the condition implies $k^2 < 4$, so $k = 1$ is the only possibility. If $n = 3m - 2$, the condition amounts to $k^2 < 2m + O(1)$, i.e., $1 \leqq k < (2m)^{1/2} + O(1)$. For such $k$, we note that $kq_{3m-2} < q_{3m+1}$ (for $m$ sufficiently large). Hence, modulo $O(m)$, we find

$$\lambda_1(q_{3m+1}) \equiv \sum_{\nu=0}^{m-1} (2\nu)^{1/2} \equiv \int_0^m (2x)^{1/2}dx \equiv {}^1/_3(2m)^{3/2}.$$

Proving the theorems now essentially amounts to obtaining $m$ as a function of $q_{3m+1}$.

PROPOSITION 2. *There exist constants* $c_1, c_2 > 0$ *such that*

$$c_1 4^m \Gamma(m + {}^3/_2) \leqq q_{3m+1} \leqq c_2\, 4^m \Gamma(m + {}^3/_2).$$

*Proof:* We note that the equations

$$q_{3m+2} = 2(m+1)q_{3m+1} + q_{3m}$$
$$q_{3m+1} = q_{3m} + q_{3m-1}$$
$$q_{3m} = q_{3m-1} + q_{3m-2}$$

may be solved to yield

$$\frac{q_{3m+1}}{q_{3m-2}} = 2(2m+1) + \frac{q_{3m-5}}{q_{3m-2}},$$

so that

$$\frac{q_{3m+1}}{q_{3m-2}} = [2(2m+1), 2(2m-1), \ldots].$$

Thus,

$$q_{3m+1} \geqq 2(2m+1)q_{3m-2} \geqq 2^2(2m+1)(2m-1)q_{3m-5} \geqq \ldots$$

and hence $q_{3m+1} \geqq c_1 4^m \Gamma(m + {}^3/_2)$ is clear. Conversely,

$$\frac{q_{3m+1}}{q_{3m-2}} \leqq 2(2m+1) + \frac{1}{2(2m-1)} = 2(2m+1)\left(1 + \frac{1}{4(2m+1)(2m-1)}\right),$$

and proceeding inductively, we see that

$$q_{3m+1} \leqq 2^m(2m+1)(2m-1)\ldots \prod_{\nu=1}^{m}\left(1 + \frac{1}{4(2\nu+1)(2\nu-1)}\right),$$

so $q_{3m+1} \leqq c_2 4^m \Gamma(m + {}^3/_2)$, where $c_2$ is determined by the infinite product.

To prove the theorems, we find to any given $B$ the integer $m$ such that $q_{3m-2} \leqq B < q_{3m+1}$. Thus,

$$c_1 4^{m-1}\Gamma(m - 1 + {}^3/_2) \leqq B < c_2 4^m \Gamma(m + {}^3/_2),$$

and

$$G(B/c_2) < m \leqq G(B/c_1) + 1.$$

Since $G(x)$ grows like $(\log x)/\log \log x$, we conclude that $m = G(B) + O(1)$, whence Theorem 1 follows at once from Proposition 1.

* This research was partially supported by National Science Foundation grant GP3990.

[1] Adams, W., and S. Lang, "Some computations in diophantine approximations," to appear in *J. Reine Angew. Math.*

[2] Khinchin, A., *Continued Fractions* (Chicago: Chicago Press, 1964).

[3] Lang, S., "Asymptotic approximations to quadratic irrationalities," *Amer. J. Math.*, **87**, 481–495 (1965).

[4] Perron, O., *Die Lehre von den Kettenbrüchen I* (Stuttgart: Teubner Verlagsgesellschaft, 1954).

[5] Schmidt, W., "A metrical theorem in diophantine approximations," *Can. J. Math.*, **11**, 619–631 (1959).

---

# *ASYMPTOTIC DIOPHANTINE APPROXIMATIONS**

BY SERGE LANG

COLUMBIA UNIVERSITY

*Communicated by S. Eilenberg, November 5, 1965*

1. *Statement of the Theorem.*—Let $\omega$, $\psi$ be positive functions of a real variable, such that $\psi(t) = \omega(t)/t$. We assume that $\psi$ is decreasing $\leqq 1$, and $\omega$ is increasing, not necessarily strictly. Let $\alpha$ be a real irrational number. We let $\lambda = \lambda_{\alpha,\psi}$ be the function such that $\lambda(B)$ is the number of solutions in integers $q$, $p$ for the inequalities

$$0 \leqq q\alpha - p < \psi(q) \quad \text{and} \quad 1 \leqq q < B, \tag{1}$$

for $B \to \infty$. We sometimes abbreviate $q\alpha - p$ by $R(q\alpha)$. Let

$$\Psi(B) = \int_1^B \psi(t)dt.$$

In reference 3, I determined $\lambda$ asymptotically when $\alpha$ is a quadratic irrationality and $\omega$ is constant, or strictly increasing satisfying some other growth condition.