*ASYMPTOTIC DIOPHANTINE APPROXIMATIONS TO E**

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1. *Statement of Results.*—Schmidt[5] proved that for almost all real numbers α, the number of solutions in integers p, q of the inequalities

$$|q\alpha - p| < 1/q \quad \text{and} \quad 1 \leqq q \leqq B$$

is asymptotic to a constant times $\log B$. One might conjecture that the classical numbers (e.g., algebraic numbers, e, π) behave like almost all numbers. Machine computations[1] were carried out for some of these numbers, and they seemed to bear out such a conjecture. Also, Lang[3] has proved that the estimate is valid when α is a real quadratic irrationality.

In this paper, we shall obtain an asymptotic estimate for e, which shows that the conjecture for e is false. The machine computations of reference 1 are misleading because the range in which they are carried out, even though going to $q \leqq 10^6$, is still too small to exhibit the proper asymptotic behavior.

The function $4^x\Gamma(x + 3/2)$ is strictly monotone-increasing. Let G be its inverse function. An application of L'Hôpital's rule shows that

$$\lim_{x\to\infty} G(x)\Bigg/\frac{\log x}{\log\log x} = 1.$$

Theorem 1. *Let $\lambda_1(B)$ be the number of solutions in integers p, q of the inequalities*

$$|qe - p| < 1/q \quad \text{and} \quad 1 \leqq q \leqq B. \tag{1}$$

Then

$$\lambda_1(B) = \frac{(2G(B))^{3/2}}{3} + O(G(B)).$$

Theorem 2.—*Let λ_2 (B) be the number of solutions in relatively prime integers p, q of the inequalities* (1) *above. Then*

$$\lambda_2(B) = 3G(B) + O(1).$$

We note that if we let $\lambda_1^+(B)$ be the number of solutions in integers p, q of the inequalities $0 < qe - p < 1/q$ and $1 \leqq q \leqq B$, then a trivial modification of the proof given below shows that $\lambda_1^+ = {}^1/_2\lambda_1$, and similarly for λ_2^+.

Theorems similar to these are true for any irrational number whose continued fraction expansion is similar to that of e. However, the notation is considerably more involved and so these extensions will be reserved for a later paper.

2. *Some Facts about Continued Fractions.*—The proofs of the theorems are based on the simple continued fraction for e. We recall some easily proved facts. See Khinchin[2] for details.

Let $\alpha > 0$ be any irrational number, and denote by $[a_0, a_1, a_2, \ldots]$ the simple continued fraction expansion of α, where the a_n are integers, positive for $n \geqq 1$. Denote the nth principal convergent to α by $p_n/q_n = [a_0, a_1, \ldots, a_n]$ and the nth

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intermediate convergents by $(p_n + rp_{n+1})/(q_n + rq_{n+1})$, where $1 \leqq r < a_{n+2}$. We shall refer to both types of convergents simply as convergents. We know that the even convergents form a strictly increasing sequence of rational numbers tending to α, and the odd convergents form a strictly decreasing sequence tending to α. Moreover, if P/Q and P'/Q' are two successive terms in either sequence, then $PQ' - P'Q = \pm 1$. We recall that $q_n = a_n q_{n-1} + q_{n-2}$, and hence, that $q_n/q_{n-1} = [a_n, a_{n-1}, \ldots, a_1]$. Further, we have always $|q_n\alpha - p_n| < 1/q_n$. Finally, let $\alpha_n = [a_n, a_{n+1}, \ldots]$. Then we have the easily derived formula

$$|(q_n + rq_{n+1})\alpha - (p_n + rp_{n+1})| = \frac{\alpha_{n+2} - r}{q_n + \alpha_{n+2}q_{n+1}}, \tag{2}$$

for integers $0 \leqq r < a_{n+2}$.

LEMMA. *If $|\alpha - p/q| < 1/q^2$ where p, q are positive integers, then p/q is a convergent of α.*

Proof: We assume that $\alpha < p/q$, the other case being proved in a similar manner. If p/q is not a convergent, then there exist two successive convergents P/Q and P'/Q' such that $\alpha < P/Q < p/q < P'/Q'$ and $P'Q - PQ' = 1$. Thus,

$$\frac{1}{q^2} > \frac{p}{q} - \alpha > \frac{p}{q} - \frac{P}{Q} \geqq \frac{1}{qQ},$$

and

$$\frac{1}{Q'q} \leqq \frac{P'}{Q'} - \frac{p}{q} < \frac{P'}{Q'} - \frac{P}{Q} = \frac{1}{Q'Q}.$$

These estimates are contradictory.

3. *Proof of the Theorem.*—We know[4] that $e = [2,1,2,1,1,4,1,1,6,1,1,8,\ldots]$ so that for e, we have $a_0 = 2$, $a_{3m} = a_{3m-2} = 1$, and $a_{3m-1} = 2m$ for all $m \geqq 1$. We set as above, $e_n = [a_n, a_{n+1}, \ldots]$.

PROPOSITION 1. *We have for all n, $m \geqq 1$,*

$$\lambda_1(q_{3m+1}) = {}^1/_3(2m)^{3/2} + O\,(m)$$

$$\lambda_2(q_n) = n + O(1).$$

Proof: We first show that, with a finite number of exceptions, the only reduced fractions p/q satisfying $|e - p/q| < 1/q^2$ are the principal convergents. From this the formula for λ_2 is clear since the nth solution is simply p_n, q_n.

By the lemma we know that any solution must be a convergent, so suppose that $(p_n + rp_{n+1})/(q_n + rq_{n+1})$ with $0 \leqq r < a_{n+2}$ is a solution. We wish to show that $r = 0$. If $n = 3m - 2$ or $n = 3m - 1$, then $a_{n+2} = 1$ and so $r = 0$. Thus, we may restrict our attention to the case where $n = 3m$. To say that this convergent is a solution simply means by (2) that

$$\frac{e_{3m+2} - r}{q_{3m} + e_{3m+2}q_{3m+1}} < \frac{1}{q_{3m} + rq_{3m+1}}.$$

This condition is equivalent to $f(r) > 0$, where

$$f(r) = r^2 + \left(\frac{q_{3m}}{q_{3m+1}} - e_{3m+2}\right) r + \left(\frac{q_{3m}}{q_{3m+1}} + e_{3m+2} - e_{3m+2}\frac{q_{3m}}{q_{3m+1}}\right).$$

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Thus, we must show that $f(1), f(2), \ldots, f(a_{3m+2} - 1) < 0$. Since f is a quadratic polynomial in r with leading coefficient 1, it suffices to show that $f(1) < 0$ and $f(a_{3m+2} - 1) < 0$. Well, $f(1) = 1 + (2 - e_{3m+2})q_{3m}/q_{3m+1}$, and $f(1) < 0$ follows from

$$e_{3m+2} > a_{3m+2} = 2(m + 1), \qquad q_{3m+1}/q_{3m} = [1,1,2m, \ldots] < 2.$$

Furthermore,

$$f(a_{3m+2} - 1) = (e_{3m+2} - a_{3m+2})\,(2 - a_{3m+2} - q_{3m}/q_{3m+1}) + 1,$$

and $f(a_{3m+2} - 1) < 0$ follows similarly.

We must now determine which multiples of the convergents p_n, q_n are also solutions of (1). Again by (2), with $r = 0$, this condition is equivalent to the condition

$$\frac{e_{n+2}}{q_n + e_{n+2}q_{n+1}} < \frac{1}{k^2 q_n} \quad \text{or} \quad k^2 < \frac{1}{e_{n+2}} + \frac{q_{n+1}}{q_n}.$$

If $n = 3m - 1$ or $n = 3m$, then the condition implies $k^2 < 4$, so $k = 1$ is the only possibility. If $n = 3m - 2$, the condition amounts to $k^2 < 2m + O(1)$, i.e., $1 \leqq k < (2m)^{1/2} + O(1)$. For such k, we note that $kq_{3m-2} < q_{3m+1}$ (for m sufficiently large). Hence, modulo $O(m)$, we find

$$\lambda_1(q_{3m+1}) \equiv \sum_{\nu=0}^{m-1} (2\nu)^{1/2} \equiv \int_0^m (2x)^{1/2}dx \equiv {}^1/_3(2m)^{3/2}.$$

Proving the theorems now essentially amounts to obtaining m as a function of q_{3m+1}.

PROPOSITION 2. *There exist constants* $c_1, c_2 > 0$ *such that*

$$c_1 4^m \Gamma(m + {}^3/_2) \leqq q_{3m+1} \leqq c_2\, 4^m \Gamma(m + {}^3/_2).$$

Proof: We note that the equations

$$q_{3m+2} = 2(m + 1)q_{3m+1} + q_{3m}$$
$$q_{3m+1} = q_{3m} + q_{3m-1}$$
$$q_{3m} = q_{3m-1} + q_{3m-2}$$

may be solved to yield

$$\frac{q_{3m+1}}{q_{3m-2}} = 2(2m + 1) + \frac{q_{3m-5}}{q_{3m-2}},$$

so that

$$\frac{q_{3m+1}}{q_{3m-2}} = [2(2m + 1), 2(2m - 1), \ldots].$$

Thus,

$$q_{3m+1} \geqq 2(2m + 1)q_{3m-2} \geqq 2^2(2m + 1)(2m - 1)q_{3m-5} \geqq \ldots$$

and hence $q_{3m+1} \geqq c_1 4^m \Gamma(m + {}^3/_2)$ is clear. Conversely,

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$$\frac{q_{3m+1}}{q_{3m-2}} \leqq 2(2m+1) + \frac{1}{2(2m-1)} = 2(2m+1)\left(1 + \frac{1}{4(2m+1)(2m-1)}\right),$$

and proceeding inductively, we see that

$$q_{3m+1} \leqq 2^m(2m+1)(2m-1)\ldots \prod_{\nu=1}^{m}\left(1 + \frac{1}{4(2\nu+1)(2\nu-1)}\right),$$

so $q_{3m+1} \leqq c_2 4^m \Gamma(m + 3/2)$, where c_2 is determined by the infinite product.

To prove the theorems, we find to any given B the integer m such that $q_{3m-2} \leqq B < q_{3m+1}$. Thus,

$$c_1 4^{m-1}\Gamma(m - 1 + 3/2) \leqq B < c_2 4^m \Gamma(m + 3/2),$$

and

$$G(B/c_2) < m \leqq G(B/c_1) + 1.$$

Since $G(x)$ grows like $(\log x)/\log\log x$, we conclude that $m = G(B) + O(1)$, whence Theorem 1 follows at once from Proposition 1.

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[2] Khinchin, A., *Continued Fractions* (Chicago: Chicago Press, 1964).

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[4] Perron, O., *Die Lehre von den Kettenbrüchen I* (Stuttgart: Teubner Verlagsgesellschaft, 1954).

[5] Schmidt, W., "A metrical theorem in diophantine approximations," *Can. J. Math.*, **11**, 619–631 (1959).

*ASYMPTOTIC DIOPHANTINE APPROXIMATIONS**

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1. *Statement of the Theorem.*—Let ω, ψ be positive functions of a real variable, such that $\psi(t) = \omega(t)/t$. We assume that ψ is decreasing $\leqq 1$, and ω is increasing, not necessarily strictly. Let α be a real irrational number. We let $\lambda = \lambda_{\alpha,\psi}$ be the function such that $\lambda(B)$ is the number of solutions in integers q, p for the inequalities

$$0 \leqq q\alpha - p < \psi(q) \quad \text{and} \quad 1 \leqq q < B, \tag{1}$$

for $B \to \infty$. We sometimes abbreviate $q\alpha - p$ by $R(q\alpha)$. Let

$$\Psi(B) = \int_1^B \psi(t)dt.$$

In reference 3, I determined λ asymptotically when α is a quadratic irrationality and ω is constant, or strictly increasing satisfying some other growth condition.

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