$$\frac{q_{3m+1}}{q_{3m-2}} \leqq 2(2m+1) + \frac{1}{2(2m-1)} = 2(2m+1)\left(1 + \frac{1}{4(2m+1)(2m-1)}\right),$$

and proceeding inductively, we see that

$$q_{3m+1} \leqq 2^m(2m+1)(2m-1)\ldots \prod_{\nu=1}^{m}\left(1+\frac{1}{4(2\nu+1)(2\nu-1)}\right),$$

so $q_{3m+1} \leqq c_2 4^m \Gamma(m + {}^3/_2)$, where c_2 is determined by the infinite product.

To prove the theorems, we find to any given B the integer m such that $q_{3m-2} \leqq B < q_{3m+1}$. Thus,

$$c_1 4^{m-1}\Gamma(m - 1 + {}^3/_2) \leqq B < c_2 4^m \Gamma(m + {}^3/_2),$$

and

$$G(B/c_2) < m \leqq G(B/c_1) + 1.$$

Since $G(x)$ grows like $(\log x)/\log\log x$, we conclude that $m = G(B) + O(1)$, whence Theorem 1 follows at once from Proposition 1.

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*ASYMPTOTIC DIOPHANTINE APPROXIMATIONS**

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1. *Statement of the Theorem.*—Let ω, ψ be positive functions of a real variable, such that $\psi(t) = \omega(t)/t$. We assume that ψ is decreasing $\leqq 1$, and ω is increasing, not necessarily strictly. Let α be a real irrational number. We let $\lambda = \lambda_{\alpha,\psi}$ be the function such that $\lambda(B)$ is the number of solutions in integers q, p for the inequalities

$$0 \leqq q\alpha - p < \psi(q) \quad \text{and} \quad 1 \leqq q < B, \tag{1}$$

for $B \to \infty$. We sometimes abbreviate $q\alpha - p$ by $R(q\alpha)$. Let

$$\Psi(B) = \int_1^B \psi(t)dt.$$

In reference 3, I determined λ asymptotically when α is a quadratic irrationality and ω is constant, or strictly increasing satisfying some other growth condition.

Schmidt[7] has shown how to remove the additional condition on ω, and he obtains an error term bounded by a constant times

$$\int_1^B \frac{\omega(t)^{1/2}}{t}\,dt$$

which is always $o(\Psi(B))$ when ω is increasing to infinity. On the other hand, Adams[1] has obtained λ asymptotically when $\alpha = e$ with $\omega = 1$, and shown that in this case, $\lambda(B)$ is asymptotic to a constant times $[(\log B)/(\log\log B)]^{2/3}$. This shows that, contrary to what happens for almost all numbers,[6] there may be an exceptional behavior for *constant* ω, and special numbers α. We shall prove, however, that as soon as ω is increasing sufficiently fast, then one recovers the expected asymptotic behavior of λ.

Let g be a positive function, assumed to be increasing. We shall say that α is of *type g* if for all sufficiently large integers B, there exists a solution in relatively prime integers q, p of the inequalities

$$0 < |q\alpha - p| < 1/q \qquad \text{and} \qquad B/g(B) \leqq q < B.$$

THEOREM. *Let α be an irrational real number of type g. Assume that ω is increasing to infinity, and that $\omega(t)^{1/2}g(t)/t$ is decreasing for all t sufficiently large. Then*

$$\lambda(B) = \Psi(B) + O\left(\int_1^B \frac{\omega(t)^{1/2}g(t)}{t}\,dt\right).$$

Remark: If h is a function decreasing to 0, then the integral from 1 to B of $\omega(t)h(t)/t$ is always $o(\Psi(B))$, so that the error term given in the theorem gives the asymptotic result $\lambda \sim \Psi$ when $g(t) \leqq \omega(t)^{1/2}h(t)$. Examples of this will now be given.

2. *Applications.*—To apply the theorem, one makes use of the type of α, which is known in a number of cases.

Example 1: Assume that g is constant. This is equivalent to saying that there exists $c > 0$ such that $|q\alpha - p| > c/q$ for all sufficiently large q. Then we can take g constant, and ω to be any increasing function to infinity. This special case was also obtained by Cassels.

Example 2: Assume that α satisfies the finiteness condition for the inequality $0 < |q\alpha - p| < 1/q^{1+\epsilon}$ for every $\epsilon > 0$. Then we can take $g(t) = t^{\epsilon}$ and the error term is $o(\Psi)$ if $\omega(t) \geqq t^{\eta}$ for some $\eta > 0$. This applies to algebraic numbers by the Thue-Siegel-Roth theorem.

Example 3: More generally, let $\{p_n/q_n\}$ be the sequence of principal convergents to α, and let f be some increasing function such that for all n sufficiently large,

$$\frac{1}{q_n f(q_n)} \leqq |q_n\alpha - p_n| \leqq \frac{1}{q_{n+1}}$$

Then $q_{n+1} \leqq q_n f(q_n)$. Given B we find n such that $q_n \leqq B \leqq q_{n+1}$. We can take $g = f$ by using the relation

$$B/f(B) \leqq B/f(q_n) \leqq q_n \leqq B.$$

Conversely, under a growth condition for g, it can be shown that $|q\alpha - p| > c/qg(q)$ for q, p relatively prime and some constant c.

As a concrete example, let $\alpha = e$. From Adams' paper, one sees that g can be taken $g(t) = c(\log t)/(\log \log t)$ for some constant c, and $\omega \leqq g/h$. In view of Adams' result, this is a best possible lower bound on ω.

Example 4: Suppose, for instance, that $f(q_n) = (\log q_n)^{1+\epsilon}$. Then we can take $g(t) = (\log t)^{1+\epsilon}$. By Khintchine's convergence theorem, this holds for almost all numbers.

We see that the theorem essentially reduces the problem of asymptotic approximation to the study of the inequality

$$0 < q\alpha - p < 1/qf(q),$$

where f is some increasing function. When f is close to 1, or grows very slowly, this is the range where the difficulties occur.

3. *Proof of the Theorem.*—In reference 7, Schmidt uses a multiplicative recursion. We shall use here an additive one.

LEMMA. *Let B, q be positive integers, satisfying the inequalities*

$$0 < |q\alpha - p| < 1/q \qquad \text{and} \qquad 1 \leqq q \leqq B/\omega(B)^{1/2},$$

and assume q, p relatively prime. Then

$$\lambda(B) - \lambda(B - q) = \int_{B-q} \psi(t)dt + \theta,$$

where $|\theta| \leqq c_1$ (and c_1 is an absolute constant).

Proof: (Cf. Behnke[2] and Ostrowski.[5]) We note that $\lambda(B) - \lambda(B - q)$ is the number of integers n satisfying

$$0 < R(n\alpha) < \psi(n) \qquad \text{and} \qquad B - q < n \leqq B. \tag{2}$$

A trivial computation shows that $0 < \psi(B - q) - \psi(B) \leqq 1/q$, whence

$$\psi(B) \leqq \psi(n) \leqq \psi(B - q) \leqq \psi(B) + \frac{1}{q}.$$

Thus, bounds for $\lambda(B) - \lambda(B - q)$ can be determined by replacing $\psi(n)$ in (2) by $\psi(B)$ and $\psi(B) + 1/q$. Since n ranges over q consecutive integers, and since $0 < \alpha - p/q < 1/q^2$, the number of solutions is equal to $q\psi(B) + O(1)$, and $q\psi(B)$ differs from the desired integral by a bounded term, thereby proving the lemma.

For B sufficiently large, select q such that

$$\frac{B}{\omega(B)^{1/2}g(B)} \leqq q < \frac{B}{\omega(B)^{1/2}} \qquad \text{and} \qquad 0 < |q\alpha - p| < 1/q.$$

Since $\omega(t)^{1/2}g(t)/t$ is decreasing, we get, using the left inequality for q,

$$\int_{B-q}^{B} \frac{\omega(t)^{1/2}g(t)}{t}\, dt \geqq \frac{q\omega(B)^{1/2}g(B)}{B} \geqq 1.$$

By the lemma, it follows that

$$\lambda(B) - \lambda(B - q) = \int_{B-q}^{B} \psi(t)dt + \theta_{B,q} \int_{B-q}^{B} \frac{\omega(t)^{1/2}g(t)}{t}\, dt,$$

with $|\theta_{B,q}| \leqq c_1$. Repeating our argument with $B - q$ instead of B, and taking the sum inductively, we find that

$$\lambda(B) = \Psi(B) + \theta\int_1^B \frac{\omega(t)^{1/2}g(t)}{t}\,dt + O(1),$$

with $|\theta| \leqq c_1$. This proves our theorem.

The same method can be used to estimate various sums. Here we limit ourselves to one example, with α of type g such that $g(t)/t$ is decreasing, namely, the sum

$$S_N = \sum_{n=1}^{N}\left(R(n\alpha) - \frac{1}{2}\right).$$

Taking $N/g(N) \leqq q < N$ and $|q\alpha - p| < 1/q$ with q, p relatively prime, we estimate $|S_N - S_{N-q}|$ and conclude inductively that

$$|S_N| = O\left(\int_1^N \frac{g(t)}{t}\,dt\right).$$

This generalizes results of references 2 and 5, and trivializes the proof.

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*INTERACTION OF NONALLELIC GENES ON CELLULAR ANTIGENS IN SPECIES HYBRIDS OF COLUMBIDAE, II. IDENTIFICATION OF INTERACTING GENES**

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Previous papers[1–6] have reported that antigenic substances can be demonstrated on the erythrocytes of certain species hybrids and backcross hybrids of Columbidae that are not detected on the cells of the parental species. These are given the term "hybrid substances" and are presumably the result of the interaction of genes. The term genic interaction implies that such an interaction took place in the chain of reactions between the respective causative genes and the end products, the cellular antigens, and does not imply a change in the genes themselves.