ERRATUM

ON THE MATHEMATICAL MECHANISM OF PHASE TRANSITION

We wish to call attention to an error in our recent note[1] which invalidates the rigorous proof of asymptotic degeneracy. We failed to notice that in certain "octants," formula (5.5) is incorrect for $0 \leq a < \eta$, i.e., in that range in which it is used in the proof (see (7.6) of ref. 1).

Let an "octant" be called ϵ-ordered ($0 < \epsilon < 1$) if the number of changes of sign in the sequence $x_1, x_2, \ldots, x_M$ of coordinates of a representative point in that octant is less than ϵM. Consider now octants which are ϵ-ordered for every ϵ ($0 < \epsilon < 1$) and such that the number of negative x's is between $M(1 - a/\eta)/2$ and $M(1 + a/\eta)/2$. In such octants, the maximum of $-\gamma q(x)$ is essentially $Mb(\nu)$ and not $M\delta(a)$ as claimed.

To repair the proof of asymptotic degeneracy, one needs to know that the integral

$$\int \varphi_1^2(\vec{x})d\vec{x}$$

extended over these "bad" octants is exponentially small (in M). This seems very likely, especially since it is true if $\varphi_1(\vec{x})$ is replaced by the approximate eigenfunction (I 3.7). However, we do not have a rigorous proof at this time.

[1] Kac, M., and C. J. Thompson, these PROCEEDINGS, 55, 676 (1966).

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[1] See, e.g., Aller, L. H., *Nuclear Transformations, Stellar Interiors, and Nebulae* (New York: Ronald Press Co., 1954), p. 182.
[2] Searle, L., A. Rodgers, W. L. W. Sargent, and J. B. Oke, *Nature*, **208**, 1190 (1965).
[3] Thackeray, A. D., *Monthly Notices Roy. Astron. Soc.*, **113**, 211 (1953).
[4] Liller, W., *Publ. Astron. Soc. Pacific*, **69**, 511 (1957).
[5] Aller, L. H., D. J. Faulkner, and R. N. Norton, *Ap. J.*, **140**, 1609 (1964); in press (1966).
[6] Hogg, A., *Photometric Observations of 244 Bright Stars*, Mount Stromlo (1958).
[7] Willstrop, R. V., *Monthly Notices Roy. Astron. Soc.*, **121**, 21 (1960).
[8] Faulkner, D. J., and L. H. Aller, *Monthly Notices Roy. Astron. Soc.*, **130**, 393 (1965).
[9] Whitford, A. E., *Astron. J.*, **63**, 201 (1958).
[10] See, e.g., Aller, L. H., *Gaseous Nebulae* (London: Chapman Hall, 1956).
[11] Burgess, A., *Monthly Notices Roy. Astron. Soc.*, **118**, 477 (1958).
[12] Clarke, W., thesis, University of California, Los Angeles (1965).
[13] Cf. Osterbrock, D. B., in *Annual Reviews of Astronomy and Astrophysics*, ed. Leo Goldberg (Palo Alto, California: Annual Reviews, Inc., 1964), vol. 1.
[14] Seaton, M. J., *Rev. Mod. Phys.*, **30**, 987 (1958).
[15] Garstang, R., *Monthly Notices Roy. Astron. Soc.*, **111**, 115 (1951).

ON THE MATHEMATICAL MECHANISM OF PHASE TRANSITION

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1. In recent years there has been a revival of interest in systems with long-range interaction. The aim has been to go beyond the classical theories of van der Waals for a gas and of Curie and Weiss for a ferromagnet. General expansions in powers of the reciprocal range of interaction have been proposed by Brout and others with the classical theory as the leading term, but all have the deficiency that they are valid only above the classical critical point. A different tack was taken by studying a class of models for which expansions could be worked out explicitly both above and below the classical critical point and also in the critical region.

Our aim here is to discuss a particular two-dimensional lattice model (model A of ref. 1). We prove the existence of a phase transition for the model and show that the mathematical mechanism which produces the transition is the same as for the two-dimensional Ising model, thus providing a bridge between the classical theories and new theories based on Onsager's solution of the two-dimensional Ising model.

2. The model we will discuss consists of a two-dimensional $N \times M$ lattice with spins $\mu_{k,\,l}$ capable of taking values $+1$ and -1 occupying the vertices of the lattice. The interaction energy in a given configuration of spins is

$$E = -\Sigma_{\substack{1 \leq k < k' \leq N \\ 1 \leq l < l' \leq M}} v(k, l; k', l')\mu_{k,l}\mu_{k',l'}, \tag{2.1}$$

where

$$v(k, l; k', l') = \alpha\gamma e^{-\gamma|k - k'|}\{{}^1/_2\delta_{l-1,l'} + {}^1/_2\delta_{l+1,l'} + \delta_{l,l'}\} \tag{2.2}$$

and, as has been shown elsewhere,[1] the free energy per spin ψ (in the thermodynamic limit), is given by

$$-\frac{\psi}{kT} = \log 2 - \frac{\nu\gamma}{2} + \lim_{M\to\infty} \frac{1}{M} \log \Lambda_{\max}, \tag{2.3}$$

where

$$\nu = \frac{\alpha}{kT} \tag{2.4}$$

and $\Lambda_{\max}$ is the maximum eigenvalue of the integral equation

$$\int \mathbf{dy}\, K(\mathbf{x}, \mathbf{y})\varphi(\mathbf{y}) = \Lambda\varphi(\mathbf{x}), \tag{2.5}$$

where we have used, and will use henceforth, the abbreviation $\mathbf{x}$ for the M-dimensional vector $(x_1, x_2, \ldots, x_M)$ and $\mathbf{dy}$ for the M-dimensional volume element $dy_1\ldots dy_M$, and where the kernel $K(\mathbf{x}, \mathbf{y})$ is defined by the formula

$$K(\mathbf{x}, \mathbf{y}) = \prod_{k=1}^{M} \left\{\cosh\sqrt{\frac{\nu\gamma}{2}}(x_k + x_{k+1})\right\}^{1/2} \prod_{k=1}^{M} \left\{\frac{W(x_k)}{W(y_k)}\right\}^{1/2} P_\gamma(x_k|y_k) \times \prod_{k=1}^{M} \left\{\cosh\sqrt{\frac{\nu\gamma}{2}}(y_k + y_{k+1})\right\}^{1/2} \tag{2.6}$$

with

$$W(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{2.7}$$

and

$$P_\gamma(x|y) = \frac{1}{\sqrt{2\pi(1 - e^{-2\gamma})}} e^{-(y-xe^{-\gamma})^2/2(1-e^{-2\gamma})}. \tag{2.8}$$

The expression (2.3) for the free energy per spin is analogous to a corresponding expression for the two-dimensional $N \times M$ Ising model,

$$-\frac{\psi}{kT} = \lim_{M\to\infty} \frac{1}{M} \log \lambda_{\max}, \tag{2.9}$$

where now $\lambda_{\max}$ is the maximum eigenvalue of a $2^M \times 2^M$ matrix V, commonly referred to as the transfer matrix.

The spectrum of V has the following properties. Above the critical point its eigenvalues are simple and their spacing is of order $1/M$. Below the critical point there are two such spectra with spacings of order $1/M$, and they are shifted with respect to each other by an amount of order $\exp(-cM)$, where c is a positive constant. It is this latter phenomenon, the *asymptotic degeneracy* of the eigenvalues, which provides the mechanism for the phase transition, and it is exactly the same mechanism which produces the transition for the model above. The spectrum of the integral equation (2.5) in fact has, at least for sufficiently small γ, exactly the same properties as that of V except that c is replaced by c/γ.

3. Before giving a proof of the asymptotic degeneracy of the largest eigenvalue of the integral equation, let us first discuss the small γ (weak long-range interaction) case, where the reason for the degeneracy will become apparent.

When γ is small, the integral equation (2.5) can be replaced, with error of order γ^2, by the differential (Schroedinger) equation (see ref. 1)

$$-\sum_{k=1}^{M} \frac{\partial^2\bar{\varphi}}{\partial x_k^2} + \bar{q}(\mathbf{x})\bar{\varphi}(\mathbf{x}) = E\bar{\varphi}(\mathbf{x}), \tag{3.1}$$

where the potential $\bar{q}(\mathbf{x})$ is given by the formula[2]

$$\bar{q}(\mathbf{x}) = {}^1/_4 \sum_{k=1}^{M} x_k^2 - \gamma^{-1} \sum_{k=1}^{M} \log \cosh \sqrt{\frac{\nu\gamma}{2}}\,(x_k + x_{k+1}) \tag{3.2}$$

and to order γ,

$$\log \Lambda_{\max} = {}^1/_2\, M\gamma - E_0\gamma, \tag{3.3}$$

where E_0 is the smallest eigenvalue of (3.1). The bar on $\bar{\varphi}(\mathbf{x})$ is used to distinguish it from the exact $\varphi(\mathbf{x})$ of (2.5).

When the quadratic part of $\bar{q}(\mathbf{x})$,

$${}^1/_4 \left\{ \sum_{k=1}^{M} x_k^2 - \nu \sum_{k=1}^{M} (x_k + x_{k+1})^2 \right\}, \tag{3.4}$$

obtained by expanding the log cosh, is positive definite, i.e., when $4\nu < 1$ (high temperatures), straightforward perturbation theory can be applied to (3.1), but when $4\nu > 1$ (low temperatures), the equation must be "tamed" before perturbation theory can be applied. The taming is performed by expanding the potential $\bar{q}(\mathbf{x})$ about its minimum which is easily seen to obtain at (see §5)

$$\pm \left(\eta\sqrt{\frac{2}{\gamma}}, \eta\sqrt{\frac{2}{\gamma}}, \ldots, \eta\sqrt{\frac{2}{\gamma}} \right), \tag{3.5}$$

where η is the nonnegative root of

$$\eta = \sqrt{4\nu}\ \tanh \sqrt{4\nu}\,\eta. \tag{3.6}$$

Above the critical point ($4\nu < 1$), (3.6) has only the trivial solution $\eta = 0$, but below the critical point ($4\nu > 1$) it has a nonzero solution. It is the fact that the potential develops two minima a distance or order $(M/\gamma)^{1/2}$ from the origin which produces the asymptotic degeneracy. To see this more clearly, note that to lowest order in γ, the eigenfunction corresponding to E_0 is

$$\frac{1}{\sqrt{2}} \left[\bar{\psi}_0\left(\mathbf{x} - \eta\sqrt{\frac{2}{\gamma}}\right) + \bar{\psi}_0\left(\mathbf{x} + \eta\sqrt{\frac{2}{\gamma}}\right) \right], \tag{3.7}$$

while the eigenfunction corresponding to the next lowest eigenvalue (E_1) is the antisymmetric form of (3.7), where $\bar{\psi}_0(\mathbf{x})$ is a normalized gaussian function corresponding to the unperturbed part of (3.1).

The overlap integral

$$\int d\mathbf{x}\, \bar{\psi}_0\left(\mathbf{x} - \eta\sqrt{\frac{2}{\gamma}}\right) \bar{\psi}_0\left(\mathbf{x} + \eta\sqrt{\frac{2}{\gamma}}\right), \tag{3.8}$$

which is a measure of the separation of E_0 and E_1, can easily be seen to be of the order $\exp(-cM/\gamma)$.[3]

One should be cautious in accepting the above argument since $\bar{\psi}_0$ is obtained from a perturbation argument and yet the $\exp(-cM/\gamma)$ factor is clearly outside the scope of perturbation theory.

4. To prove rigorously, for sufficiently small γ, that the largest eigenvalue of the integral equation is asymptotically degenerate, we first order the eigenvalues in decreasing order

$$\Lambda_{\max} = \Lambda_1 \geq \Lambda_2 \geq \ldots,$$

and denote their corresponding normalized eigenfunctions by $\varphi_1, \varphi_2, \ldots$, respectively. Then since $\varphi_1(\mathbf{x})$ is symmetric under $\mathbf{x} \to -\mathbf{x}$ (which follows simply from the symmetry of $K(\mathbf{x}, \mathbf{y})$ under $\mathbf{x} \to -\mathbf{x}$ and $\mathbf{y} \to -\mathbf{y}$), a suitable trial function for $\varphi_2(\mathbf{x})$ (in the sense that it is orthogonal to $\varphi_1(\mathbf{x})$) is the antisymmetric function

$$\psi(\mathbf{x}) = \varphi_1^+(\mathbf{x}) - \varphi_1^-(\mathbf{x}), \tag{4.1}$$

where

$$\varphi_1^{\pm}(\mathbf{x}) = \begin{cases} \varphi_1(\mathbf{x}) & \text{if } u(\mathbf{x}) \gtrless 0, \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

and $u(\mathbf{x})$ is defined by the formula

$$u(\mathbf{x}) = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} x_k. \tag{4.3}$$

Thus, since $\varphi_1(\mathbf{x}) = \varphi_1^+(\mathbf{x}) + \varphi_1^-(\mathbf{x})$,

$$\begin{aligned} \Lambda_2 &\geq \int d\mathbf{x} \int d\mathbf{y} \psi(\mathbf{x}) K(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) \\ &= \Lambda_1 - 4 \int d\mathbf{x} \int d\mathbf{y} \varphi_1^+(\mathbf{x}) K(\mathbf{x}, \mathbf{y}) \varphi_1^-(\mathbf{y}). \end{aligned} \tag{4.4}$$

To estimate the overlap integral $(\varphi_1^+, K\varphi_1^-)$ on the right-hand side of (4.4), we first write the kernel $K(\mathbf{x}, \mathbf{y})$ in the form

$$K(\mathbf{x}, \mathbf{y}) = \exp\left(-\frac{\gamma}{2} q(\mathbf{x})\right) F(\mathbf{x}, \mathbf{y}) \exp\left(-\frac{\gamma}{2} q(\mathbf{y})\right), \tag{4.5}$$

where

$$\frac{\gamma}{2} q(\mathbf{x}) = {}^1/_4 \tanh\left(\frac{\gamma}{2}\right) \sum_{k=1}^{M} x_k^2 - {}^1/_2 \sum_{k=1}^{M} \log \cosh \sqrt{\frac{\nu\gamma}{2}} (x_k + x_{k+1}) \tag{4.6}$$

and

$$F(\mathbf{x}, \mathbf{y}) = \prod_{k=1}^{M} \exp\left(-\frac{(y_k - x_k)^2}{4 \sinh \gamma}\right) / [2\pi(1 - e^{-2\gamma})]^{1/2}. \tag{4.7}$$

Using Schwarz's inequality, we obtain

$$\begin{aligned} (\varphi_1^+, K\varphi_1^-) &= \int_{u(\mathbf{x}) > 0} d\mathbf{x} \int_{u(\mathbf{y}) < 0} d\mathbf{y} \left\{ \varphi_1(\mathbf{x}) e^{-\frac{\gamma}{2} q(\mathbf{x})} \sqrt{F(\mathbf{x}, \mathbf{y})} \right\} \\ &\quad \times \left\{ \varphi_1(\mathbf{y}) e^{-\frac{\gamma}{2} q(\mathbf{y})} \sqrt{F(\mathbf{x}, \mathbf{y})} \right\} \\ &\leq \int_{u(\mathbf{x}) > 0} d\mathbf{x}\, \varphi_1^2(\mathbf{x}) e^{-\gamma q(\mathbf{x})} \int_{u(\mathbf{y}) < 0} d\mathbf{y}\, F(\mathbf{x}, \mathbf{y}). \end{aligned} \tag{4.8}$$

The integral over $\mathbf{y}$ can be performed straightforwardly with the result that

$$\int_{\substack{u(\mathbf{x}) > 0 \\ u(\mathbf{y}) < 0}} d\mathbf{y}\, F(\mathbf{x}, \mathbf{y}) = \frac{\exp(M\gamma/2)}{\sqrt{2\pi}} \int_{u(\mathbf{x})/(2 \sinh \gamma)^{1/2}}^{\infty} e^{-t^2/2} dt \leq \exp(M\gamma/2) \sqrt{e}\, e^{-\frac{u^2(\mathbf{x})}{4 \sinh \gamma}}. \tag{4.9}$$

Finally,

$$(\varphi_1^+, K\varphi_1^-) < \sqrt{e}\exp(M\gamma/2)\int d\mathbf{x}\varphi_1^2(\mathbf{x})e^{-\gamma q(\mathbf{x})}e^{-\frac{u^2(\mathbf{x})}{4\sinh\gamma}}. \tag{4.10}$$

5. Let us now notice that

$$\begin{aligned}
-\gamma q(\mathbf{x}) &= -{}^1/_2\left(\tanh\frac{\gamma}{2}\right)\sum_{k=1}^{M} x_k^2 + \sum_{k=1}^{M}\log\cosh\sqrt{\frac{\nu\gamma}{2}}(x_k + x_{k+1}) \\
&= -{}^1/_8\left(\tanh\frac{\gamma}{2}\right)\sum_{k=1}^{M}[(x_k + x_{k+1})^2 + (x_k - x_{k+1})^2] \\
&\qquad + \sum_{k=1}^{M}\log\cosh\sqrt{\frac{\nu\gamma}{2}}(x_k + x_{k+1}) \\
&\leq -{}^1/_8\left(\tanh\frac{\gamma}{2}\right)\sum_{k=1}^{M}(x_k + x_{k+1})^2 + \sum_{k=1}^{M}\log\cosh\sqrt{\frac{\nu\gamma}{2}}(x_k + x_{k+1})
\end{aligned}$$

and that equality obtains if all x's are equal.

Clearly,

$$-\gamma q(\mathbf{x}) \leq M\max_y\left\{-{}^1/_8\left(\tanh\frac{\gamma}{2}\right)y^2 + \log\cosh\sqrt{\frac{\nu\gamma}{2}}y\right\} = Mb(\nu), \tag{5.1}$$

where

$$b(\nu) = -\frac{1}{\gamma}\left(\tanh\frac{\gamma}{2}\right)\bar{\eta}^2 + \log\cosh 2\sqrt{\nu}\,\bar{\eta} \tag{5.2}$$

and $\bar{\eta}$ is the nonnegative solution of the equation

$$\left(\frac{2}{\gamma}\tanh\frac{\gamma}{2}\right)\bar{\eta} = 2\sqrt{\nu}\tanh 2\sqrt{\nu}\,\bar{\eta}, \tag{5.3}$$

which, in the limit $\gamma \to 0$, becomes (3.6).

Thus $Mb(\nu)$ is the maximum of $-\gamma q(\mathbf{x})$, achieved when either all x_j are equal to $\bar{\eta}\sqrt{\frac{2}{\gamma}}$ or to $-\bar{\eta}\sqrt{\frac{2}{\gamma}}$.

If

$$\nu > \frac{1}{2\gamma}\tanh\left(\frac{\gamma}{2}\right), \tag{5.4}$$

$\bar{\eta}$ is positive and so is $b(\nu)$. In addition, $b(\nu)$ is an increasing function of ν.

We make two additional observations which will be needed in the sequel.

(*a*) The maximum of $-\gamma q(\mathbf{x})$ subject to the constraint

$$u(\mathbf{x}) = \frac{1}{\sqrt{M}}(x_1 + \ldots + x_M) = a\sqrt{\frac{2M}{\gamma}}$$

is easily seen to be

$$M\delta(a) = M\left\{-\frac{1}{\gamma}\left(\tanh\frac{\gamma}{2}\right)a^2 + \log\cosh 2\sqrt{\nu}\,a\right\}. \tag{5.5}$$

An equivalent way of stating this result is the inequality

$$-\gamma q(\mathbf{x}) \leq -\,{}^1/_2\left(\tanh\frac{\gamma}{2}\right)u^2(\mathbf{x}) + M\log\cosh\sqrt{\frac{2\nu\gamma}{M}}\,u(\mathbf{x}). \tag{5.6}$$

(*b*) We have, for all $\mathbf{x}$,

$$-\gamma q(\mathbf{x}) \geq Mb(\nu) - {}^1/_2\left(\tanh\frac{\gamma}{2}\right)\sum_{k=1}^{M}\left(x_k - \bar{\eta}\sqrt{\frac{2}{\gamma}}\right)^2. \tag{5.7}$$

This can be proved by noting that

$$\begin{aligned} -\gamma q(\mathbf{x}) + {}^1/_2\left(\tanh\frac{\gamma}{2}\right)\sum_{k=1}^{M}\left(x_k - \bar{\eta}\sqrt{\frac{2}{\gamma}}\right)^2 \\ = \frac{M}{\gamma}\left(\tanh\frac{\gamma}{2}\right)\bar{\eta}^2 - \frac{1}{2}\left(\tanh\frac{\gamma}{2}\right)\bar{\eta}\sqrt{\frac{2}{\gamma}}\sum_{k=1}^{M}(x_k + x_{k+1}) \\ + \sum_{k=1}^{M}\log\cosh\sqrt{\frac{\nu\gamma}{2}}\,(x_k + x_{k+1}) \end{aligned}$$

achieves its *minimum* $Mb(\nu)$ for

$$x_1 = x_2 = \ldots = x_M = \bar{\eta}\sqrt{\frac{2}{\gamma}}.$$

6. Inequality (5.7) permits one to obtain a lower bound for Λ_1 which will be useful a little later on.

Let

$$f(\mathbf{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{M}{2}} e^{-\frac{1}{4}\sum_{k=1}^{M}\left(x_k - \bar{\eta}\sqrt{\frac{2}{\gamma}}\right)^2} \tag{6.1}$$

so that

$$\int d\mathbf{x}\, f^2(\mathbf{x}) = 1.$$

By the Raleigh-Ritz principle,

$$\Lambda_1 \geq \int\int d\mathbf{x}\, d\mathbf{y}\, f(\mathbf{x})\, K(\mathbf{x},\mathbf{y})\, f(\mathbf{y}),$$

and hence by (5.7),

$$\Lambda_1 \geq e^{Mb(\nu)}\int\int d\mathbf{x}\, d\mathbf{y}\, g(\mathbf{x})\, F(\mathbf{x},\mathbf{y})\, g(\mathbf{y}), \tag{6.2}$$

where

$$g(\mathbf{x}) = f(\mathbf{x})e^{-\frac{1}{2}\left(\tanh\frac{\gamma}{2}\right)\sum_{k=1}^{M}\left(x_k - \bar{\eta}\sqrt{\frac{2}{\gamma}}\right)^2}.$$

The integral in (6.2) is elementary and one obtains easily

$$\Lambda_1 \geq e^{M(b(\nu) - \bar{c}(\gamma))}, \tag{6.3}$$

where

$$\bar{c}(\gamma) = {}^1/_2\left\{\gamma + \log\left(1 + \tanh\frac{\gamma}{2}\right) + \log\left[1 + {}^1/_2\sinh\gamma\left(1 + \tanh\frac{\gamma}{2}\right)\right]\right\}. \tag{6.4}$$

7. We now go back to the inequality (4.10) and use (5.6). This yields

$$(\varphi_1^+, K\varphi_1^-) \leq \sqrt{e}\, e^{\frac{M\gamma}{2}} \int d\mathbf{x}\varphi_1^2(\mathbf{x}) \exp\left\{- {}^1/_2\left(\tanh \gamma + \frac{1}{2 \sinh \gamma}\right) u^2(\mathbf{x}) + M \log \cosh\left(\sqrt{\frac{2\nu\gamma}{M}}\, u(\mathbf{x})\right)\right\}. \quad (7.1)$$

Let us begin by assuming that

$$4 \sinh \gamma < 1, \quad (7.2)$$

and hence the exponent in the integrand of (7.1) is smaller than[4]

$$-u^2(\mathbf{x}) + M \log \cosh\left(\sqrt{\frac{2\nu\gamma}{M}}\, u(\mathbf{x})\right) < - \frac{u^2(\mathbf{x})}{2} + M\nu\gamma. \quad (7.3)$$

Estimate (7.1) now implies that

$$(\varphi_1^+, K\varphi_1^-) < \sqrt{e}\, e^{M\gamma\left(\nu + \frac{1}{2}\right)} \int d\mathbf{x}\, \varphi_1^2(\mathbf{x}) e^{-\frac{u^2(\mathbf{x})}{2}}. \quad (7.4)$$

To estimate the integral in (7.4), we multiply the integral equation (2.5) by $\varphi_1(\mathbf{x}) \exp\{-u^2(\mathbf{x})/2\}$, integrate both sides, and apply Schwarz's inequality. We thus obtain

$$\int d\mathbf{x}\, \varphi_1^2(\mathbf{x})\, e^{-\frac{u^2(\mathbf{x})}{2}} = \frac{1}{\Lambda_1} \iint d\mathbf{x}\, d\mathbf{y} \left\{\varphi_1(\mathbf{x}) e^{-\frac{\gamma}{2} q(\mathbf{x})}\, e^{-\frac{1}{4} u^2(\mathbf{x})} \sqrt{F(\mathbf{x}, \mathbf{y})}\right\} \times \left\{\varphi_1(\mathbf{y}) e^{-\frac{\gamma}{2} q(\mathbf{y})}\, e^{-\frac{1}{4} u^2(\mathbf{x})} \sqrt{F(\mathbf{x}, \mathbf{y})}\right\}$$
$$\leq e^{-M\left(b(\nu) - \bar{c}(\gamma) - \frac{\gamma}{2}\right)} \int d\mathbf{x}\varphi_1^2(\mathbf{x}) e^{-\frac{u^2(\mathbf{x})}{2}}\, e^{-\gamma q(\mathbf{x}) + \sinh\gamma\, u^2(\mathbf{x})}, \quad (7.5)$$

where in addition to Schwarz's inequality, we have used the bound (6.3) for Λ_1, and the facts that

$$\int d\mathbf{y} F(\mathbf{x}, \mathbf{y}) = e^{\frac{M\gamma}{2}} \leq e^{\frac{M\gamma}{2} + \sinh\gamma\, u^2(\mathbf{y})}$$

and

$$\int d\mathbf{x} F(\mathbf{x}, \mathbf{y})\, e^{-\frac{u^2(\mathbf{x})}{2}} = \frac{e^{\frac{M\gamma}{2}}}{\sqrt{1 + 2 \sinh \gamma}}\, e^{-\frac{u^2(\mathbf{y})}{2(1 + 2 \sinh\gamma)}} < e^{\frac{M\gamma}{2}}\, e^{-\frac{u^2(\mathbf{y})}{2} + \sinh\gamma\, u^2(\mathbf{y})}.$$

Now from (5.5), we have, when $|u(\mathbf{x})| < \epsilon\sqrt{\frac{2M\nu}{\gamma}}$,

$$-M\left\{b(\nu) - \bar{c}(\gamma) - \frac{\gamma}{2}\right\} - \gamma q(\mathbf{x}) + \sinh \gamma u^2(\mathbf{x}) \leq$$
$$M\left\{b(\nu) - \bar{c}(\gamma) - \frac{\gamma}{2}\right\} + M\left(\frac{2}{\gamma} \sinh \gamma - \frac{1}{\gamma} \tanh \frac{\gamma}{2}\right) \epsilon^2\nu + 2\epsilon\nu M = -M\sigma. \quad (7.6)$$

And since for large ν, $b(\nu)$ is proportional to ν (with a coefficient of proportionality which depends on γ, but insensitively if γ is limited by (7.2)), it is clear that if ϵ is chosen sufficiently small (independently of γ in the range (7.2)), then for all $\nu > \bar{\nu}$, where $\bar{\nu}$ is sufficiently large, $\sigma > 0$.

If we now divide the ranges of integration in (7.5) up into $|u(\mathbf{x})| < \epsilon\sqrt{\frac{2M\nu}{\gamma}}$ and $|u(\mathbf{x})| \geq \epsilon\sqrt{\frac{2M\nu}{\gamma}}$ with ϵ chosen as above, we have, on rearranging,

$$\begin{aligned}(1 - e^{-M\sigma}) \int d\mathbf{x}\varphi_1{}^2(\mathbf{x})e^{-\frac{u^2(\mathbf{x})}{2}} &\leq e^{-M\left(b(\nu) - \bar{c}(\gamma) - \frac{\gamma}{2}\right)} \int_{|u(\mathbf{x})| \geq \epsilon\sqrt{\frac{2M\nu}{\gamma}}} d\mathbf{x}\varphi_1{}^2(\mathbf{x})\, e^{-\frac{u^2(\mathbf{x})}{2}} e^{-\gamma q(\mathbf{x}) + \sinh\gamma\, u^2(\mathbf{x})} \\ &\leq e^{M\left(\bar{c}(\gamma) + \frac{\gamma}{2}\right)} e^{-\left(\frac{1}{2} - \sinh\gamma\right)\frac{2M\epsilon^2\nu}{\gamma}} \int_{|u(\mathbf{x})| > \epsilon\sqrt{\frac{2M\nu}{\gamma}}} d\mathbf{x}\varphi_1{}^2(\mathbf{x}) \leq e^{M\left(\bar{c}(\gamma) + \frac{\gamma}{2}\right)} e^{-\frac{M\epsilon^2\nu}{2\gamma}}, \end{aligned} \tag{7.7}$$

where in the last step, use has been made of (7.2) and of the fact that φ_1 is normalized. Dividing through by $1 - e^{-M\sigma}$ (which is positive) and substituting back into (7.4) yields

$$(\varphi_1{}^+, K\varphi_1{}^-) < \sqrt{e}\,(1 - e^{-M\sigma})^{-1} e^{M(\bar{c}(\gamma) + \gamma + \nu\gamma)} e^{-\frac{M\epsilon^2\nu}{2\gamma}}. \tag{7.8}$$

And if γ is now chosen, so that in addition to (7.2) one has

$$\frac{\epsilon^2}{2\gamma} > \gamma \tag{7.9}$$

and then if ν is again chosen sufficiently large, we have finally

$$(\varphi_1{}^+, K\varphi_1{}^-) < e^{-\frac{cM}{\gamma}}, \tag{7.10}$$

where c is a positive constant. This, together with (4.4), proves that the maximum eigenvalue is asymptotically degenerate.

8. Our proof of asymptotic degeneracy (which implies setting in of long-range order) suffers from the fault that it holds only for sufficiently small γ. This deficiency, however, which will no doubt be remedied by better ways of estimating for moderate or large γ, is not very serious.

What is important is that, unlike in previous one-dimensional models, one no longer needs the limit $\gamma \to 0$ to produce a phase change. Through the appearance of M in $\exp(-cM/\gamma)$ we achieve the transition because we must let $M \to \infty$, which is part of taking the *thermodynamic limit.* Smallness of γ is now a matter of convenience, not necessity.

[1] Kac, M., and E. Helfand, *J. Math. Phys.*, **4**, 1078 (1963).

[2] It is understood here as well as in the sequel that $x_{M+1} = x_1$.

[3] That the actual separation is of this order was first conjectured on somewhat different grounds by E. Helfand [see, e.g., Frisch, H. L., and J. L. Lebowitz, *The Equilibrium Theory of Classical Fluids* (New York: W. A. Benjamin, Inc., 1965), pp. III-64].

[4] We use here the fact log cosh $\zeta \leqslant |\zeta|$.