FURTHER REMARKS ON NONLINEAR P-COMPACT OPERATORS IN BANACH SPACE

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1. *Introduction.*—In references 8 and 9, the author derived a number of results concerning the solution of nonlinear equations involving bounded projectionally compact (P-compact) operators defined on a real Banach space with a basis. These results extended and simplified similar results for quasicompact operators obtained by Kaniel.[5] At the same time we deduced from our theorems certain basic results on bounded monotone operators obtained previously by Minty,[7] Browder,[1, 2] and others.

The purpose of this note is twofold: First, we extend our main results in references 8 and 9 to P-compact operators without assuming their boundedness; second, we generalize to our class of operators the theorems of Granas[3, 4] concerning the solvability of nonlinear equations and the geometrical intersection theorem involving quasi-bounded completely continuous operators. At the same time we obtain some new results for P-compact and monotone operators in Hilbert space. (The expanded version of this note with detailed proofs will appear elsewhere.)

2. *Extended Results for P-Compact Operators.*—Let X be a real Banach space with the property that there exists a sequence $\{X_n\}$ of finite dimensional subspaces X_n of X, a sequence of linear projections $\{P_n\}$ defined on X, and a constant $K > 0$ such that

$$P_nX = X_n, \qquad X_n \subset X_{n+1}, \qquad n = 1, 2, 3, \ldots, \qquad \overline{\bigcup_n^{\infty} X_n} = X, \tag{1}$$

$$\|P_n\| \leq K, \qquad n = 1, 2, 3, \ldots \tag{2}$$

Let B_r denote the closed ball in X of radius $r > 0$ about the origin and S_r the boundary of B_r. Let the symbols $\to$ and $\rightharpoonup$ denote the strong and weak convergence in X, respectively.

Definition 1: A nonlinear operator A mapping X into itself is called *Projectionally compact* (*P-compact*) if P_nA is continuous in X_n for all sufficiently large n and if for any constant $p > 0$ and any bounded sequence $\{x_n\}$ with $x_n \in X_n$ the strong convergence of the sequence $\{g_n\} \equiv \{P_nAx_n - px_n\}$ implies the existence of a strongly convergent subsequence $\{x_{n_i}\}$ and an element x in X such that $x_{n_i} \to x$ and $P_{n_i}Ax_{n_i} \to Ax$.

The results of this paper are based on the following theorem which for bounded P-compact operators was proved by the author in references 8 and 9.

Theorem 1. *Suppose that A is P-compact. Suppose further that for given $r > 0$ and $\mu > 0$ the operator A satisfies both of the following conditions:*

(Λ): *There exists a number $c(r) > 0$ such that if, for any n, $P_nAx = \lambda x$ holds for x in S_r with $\lambda > 0$, then $\lambda \leq c(r)$.*

($\Pi\mu$): *If for some x in S_r the equation $Ax = \alpha x$ holds, then $\alpha < \mu$.*

Then there exists an element u in $(B_r - S_r)$ such that

$$Au - \mu u = 0. \tag{3}$$

The proof of Theorem 1 follows the same line of argument as the proof of the corresponding Theorem 2 in reference 9.

Remark 1: Let us remark that condition (Λ) is in no way a condition on the size of Ax or even on the size of P_nAx. All it says is that when for any x in S_r and any n the vector P_nAx is in the same direction as x, then P_nAx are uniformly bounded.

Remark 2: The assertion of Theorem 1 remains valid if condition (Λ) is replaced, for example, by any one of the following stronger conditions:

(*a*) A is bounded, i.e., A maps bounded sets in X into bounded sets.

(*b*) For any given $r > 0$, the set $A(S_r)$ is bounded.

(*c*) X is a Hilbert space H and, for any given $r > 0$, $(Ax, x) \leq c\|x\|^2$ for all x in S_r and some $c > 0$.

COROLLARY 1. *If A is P-compact and for some $r > 0$ the conditions* (Λ) and (Π_1) *are satisfied on S_r, then A has a fixed point in $(B_r - S_r)$.*

In Theorem 2 below we consider the problem of solving the equation

$$Au - \mu u = f, \qquad f \in X, \tag{4}$$

where f is any given element in X, $\mu > 0$, and A is P-compact. New results for monotone and quasi-bounded operators will be deduced from Theorem 2.

THEOREM 2. *Suppose that A is P-compact. Suppose further that there exists a sequence of spheres $\{S_{r_p}\}$ with $r_p \to \infty$, as $p \to \infty$, and two sequences of positive numbers $c_p = c(r_p)$ and $k_p = k(r_p)$ with $k_p \to \infty$, as $r_p \to \infty$, such that the following conditions hold:*

(Λ_f): *Whenever for any given f in B_{k_p} and any n the equation $P_nAx - \lambda x = P_nf$ holds for x in S_{r_p} with $\lambda > 0$, then $\lambda \leqslant c_p$.*

(Π_p): $\|Ax - \eta x\| \geqslant k_p$ *for any $\eta \geqslant \mu > 0$ and any x in S_{r_p}.*

Then for every f in X there exists a u in X which satisfies equation (4).

For Hilbert space H we deduce from Theorem 2 the following theorem from which, in turn, we obtain a new result for monotone operators.

THEOREM 3. *If A is a P-compact mapping of H into itself such that*

$$(Ax, x) \leq (A(0), x), \qquad x \in H, \tag{5}$$

then for any given $\mu > 0$ the operator $(A - \mu I)$ is onto.

COROLLARY 2. *If A is P-compact and monotone decreasing, i.e.,*

$$(Ax - Ay, x - y) \leq 0, \qquad x, y \in H, \tag{6}$$

then for any given $\mu > 0$ the mapping $P = A - \mu I$ is one-to-one and onto.

3. *Applications to Quasi-Bounded Mappings.*—Let Y and Z be any two real Banach spaces. Following Granas[3] we say that a nonlinear mapping A of Y into Z is *quasi-bounded* if there exist two constants $M > 0$ and $q_0 > 0$ such that

$$\|Ax\| \leq M\|x\| \text{ for all } x \text{ in } Y \text{ with } \|x\| \geq q_0. \tag{7}$$

If A is quasi-bounded, then the number $|A|$ defined by

$$|A| = \inf_{q_0 \leq q < \infty} \left\{ \sup_{\|x\| \geq q} \frac{\|Ax\|}{\|x\|} \right\} \tag{8}$$

is called the *quasi-norm* of A. Let us note in passing that every nonlinear mapping of Y into Z which is asymptotically differentiable in the sense of Krasnoselsky[6] is quasi-bounded.

In this section we apply our Theorems 1 and 2 to the generalization of theorems obtained by Granas[3, 4] for completely continuous and quasi-bounded operators.

THEOREM 4. *Suppose that A is a P-compact and quasi-bounded mapping of X into itself. If $\mu > M$, then $(A - \mu I)$ is onto.*

Remark 3: It is not hard to see that Theorem 4 remains valid if instead of assuming that $\mu > M$, we assume that $\mu > |A|$.

COROLLARY 3. *Suppose that A is quasi-bounded and P-compact with $p < 0$. If $\mu > M$, then $(\mu I + A)$ maps X onto itself.*

Remark 4: When A is completely continuous (i.e., A is continuous and compact) and $\mu = 1$, Corollary 3 was proved by Granas[3] by the application of the topological fixed-point theorem of Rothe.[10]

An intersection theorem in X: Suppose that X is a direct sum of the subspaces $V \subset X$ and $W \subset X$, i.e., $X = V \oplus W$, and suppose that P_V and P_W denote the projections of X onto V and W, respectively. It is obvious that P_V and P_W are linear and that

$$||P_V x|| \leq ||P_V||\,||x||, \qquad ||P_W x|| \leq ||P_W||\,||x||, \qquad x \in X. \tag{9}$$

Suppose that $f(v) = v + F(v)$ maps V into X and that $g(w) = w + G(w)$ maps W into X, where F maps V into X and G maps W into X. Using Rothe's theorem, Granas[4] obtained an interesting intersection theorem by proving that if F and G are quasi-bounded and completely continuous and if

$$|F|\,||P_V|| + |G|\,||P_W|| < 1, \tag{10}$$

then the images $f(V)$ and $g(W)$ have a nonempty intersection, i.e., $f(V) \cap g(W) \neq \emptyset$.

Here we consider the intersection theorem when either F or G is P-compact and when condition (10) is replaced by a weaker condition. Our result is based on the application of Theorem 1.

THEOREM 5. *Let G be a nonlinear mapping of W into X such that the operator $G(-P_W)$ is P-compact and such that to a given $r > 0$ there corresponds a number $c(r) > 0$ with the property that for all x in S_r*

$$||G(-P_W x)|| \leq c(r). \tag{11}$$

Let F be a completely continuous nonlinear mapping of V into X and let $f_\mu(v)$ and $g_\mu(w)$ be the mappings defined, respectively, from V and W to X by $f_\mu(v) = \mu v + Fv$ and $g_\mu(w) = \mu w + Gw$. Suppose that for given $r > 0$ and $\mu > 0$ the operators F and G satisfy the condition

(II): *If $Fv + \alpha v = Gw + \alpha w$ for some v in V and w in W with $||v - w|| = r$, then $\alpha < \mu$.*

Then $f_\mu(V) \cap g_\mu(W) \neq \emptyset$.

COROLLARY 4. *Suppose that G and F satisfy all conditions of Theorem 5 except that condition* (II) *is replaced by the condition*

$$||\mu w + Gw - (\mu v + Fv)||^2 \geq ||Fv - Gw||^2 - \mu^2||v - w||^2 \text{ for } v \in V, \; w \in W \text{ with } ||v - w|| = r. \tag{12}$$

Then $f_\mu(V) \cap g_\mu(W) \neq \emptyset$.

Remark 5: In case X is a Hilbert space, (12) is equivalent to

$$(G(-P_W x) - F(P_V x), x) \leq ||x||^2, \qquad x \in S_r. \tag{13}$$

Let us note that the main result of Granas[4] is an immediate consequence of our Theorem 5 above and Theorem 3 in reference 9.

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[4] *Ibid.*, "On a geometrical theorem in Banach space," pp. 973–877.

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THE CLASSIFICATION OF THE COMPLEX PRIMITIVE INFINITE PSEUDOGROUPS

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The primitive infinite pseudogroups were first classified by E. Cartan;[1] however, there were some serious gaps in his proof (cf. refs. 5 or 6). In this note we sketch a complete proof of the classification theorem. This proof follows Cartan's quite closely; however, Lemmas 1 and 2 below lead to essential simplifications in his proof in addition to supplying the missing details.

Like Cartan, we only consider *complex analytic* pseudogroups. Thus the underlying manifold has a complex structure, the local diffeomorphisms are holomorphic, and the differential equations defining the pseudogroup are complex analytic. For definitions and elementary facts about pseudogroups, we refer to references 4 or 6.

Definition: A pseudogroup, Γ, which acts on a manifold M, is called *primitive* if there is no (nontrivial) foliation on M whose leaves are permuted by Γ.

A primitive pseudogroup is necessarily transitive, otherwise its orbits would be an invariant foliation.

Let Γ be a primitive pseudogroup acting on the manifold M. Let x_0 be a fixed point of M, let T_{x_0} be the tangent space to M at x_0, let $T^*_{x_0}$ be the dual space, and