Then $f_\mu(V) \cap g_\mu(W) \neq \emptyset$.

Remark 5: In case X is a Hilbert space, (12) is equivalent to

$$(G(-P_W x) - F(P_V x), x) \leq \|x\|^2, \qquad x \in S_r. \tag{13}$$

Let us note that the main result of Granas[4] is an immediate consequence of our Theorem 5 above and Theorem 3 in reference 9.

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THE CLASSIFICATION OF THE COMPLEX PRIMITIVE INFINITE PSEUDOGROUPS

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The primitive infinite pseudogroups were first classified by E. Cartan;[1] however, there were some serious gaps in his proof (cf. refs. 5 or 6). In this note we sketch a complete proof of the classification theorem. This proof follows Cartan's quite closely; however, Lemmas 1 and 2 below lead to essential simplifications in his proof in addition to supplying the missing details.

Like Cartan, we only consider *complex analytic* pseudogroups. Thus the underlying manifold has a complex structure, the local diffeomorphisms are holomorphic, and the differential equations defining the pseudogroup are complex analytic. For definitions and elementary facts about pseudogroups, we refer to references 4 or 6.

Definition: A pseudogroup, Γ, which acts on a manifold M, is called *primitive* if there is no (nontrivial) foliation on M whose leaves are permuted by Γ.

A primitive pseudogroup is necessarily transitive, otherwise its orbits would be an invariant foliation.

Let Γ be a primitive pseudogroup acting on the manifold M. Let x_0 be a fixed point of M, let T_{x_0} be the tangent space to M at x_0, let $T^*_{x_0}$ be the dual space, and let $S^k(T^*)_{x_0}$ be its k-fold symmetric product. Let γ_k be the set of Γ-vector fields (cf. ref. 6) which vanish to order k at x_0. The quotient γ_{k-1}/γ_k can be identified with a subspace, g^k, of $T_{x_0} \otimes S^k(T^*)_{x_0}$; and, in particular, γ_0/γ_1 can be identified with a subalgebra, g, of $\mathrm{Hom}(T_{x_0},T_{x_0}) \approx T_{x_0} \otimes T^*_{x_0}$. This subalgebra is called the *linear isotropy algebra* of Γ at x_0. Our first task is to determine it completely.

We make the following definition:

A nonzero vector, $\xi \epsilon T^*_{x_0}$, is *characteristic* if for every positive integer k there exists a $v \epsilon T_x$ such that $v \otimes \xi^k \epsilon g^k$.

LEMMA 1. *A necessary and sufficient condition that* Γ *be of infinite type is that non-characteristic vectors exist.*

In reference 2 we prove the following:

LEMMA 1′. *A necessary and sufficient condition that* g *be of infinite type is that it contain elements of rank one.*

Lemma 1 follows by applying Lemma 1′ to Γ and its various prolongations.

We will say that a characteristic vector $\xi \epsilon T^*_{x_0}$ is *integrable* if there exists a neighborhood $\mathfrak{U}$ of x_0 and a (holomorphic) function f defined in $\mathfrak{U}$ such that $(df)_{x_0} = \xi$ and $(df)_x$ is characteristic for all x in $\mathfrak{U}$.

LEMMA 2. *If* Γ *is infinite, there exist integrable characteristic vectors.*

The proof of this fact depends on results in partial differential equations which will be discussed elsewhere (cf. ref. 3).

COROLLARY. *The set of vectors,* ω, *in* $T^*_{x_0}$ *with the property that* $\ni v \epsilon T_{x_0}$, *with* $v \otimes \omega \epsilon g$, *spans* $T^*_{x_0}$.

Proof: Let C_x be the subspace of T^*_x spanned by the integrable characteristic vectors. The set of integrable characteristic vectors is mapped onto itself by Γ; therefore, since Γ is transitive, dim C_x does not vary from point to point. Thus $x \rightarrow C_x$ gives rise to an invariant foliation on M. By the lemma, dim $C_x > 0$; therefore, since Γ is primitive, $C_x = T^*_x$ and in particular $C_{x_0} = T^*_{x_0}$. q.e.d.

Let V be a subspace of T_{x_0} invariant under g, and let W be the quotient, T_{x_0}/V. There is a natural Lie algebra homomorphism $g \rightarrow \mathrm{Hom}(W,W)$ whose kernel we will denote by g_1.

LEMMA 4. $g|V$ *and* $g_1|V$ *are both of infinite type.*

Proof: By Lemma 3 there exist elements of rank one in g whose restrictions to V do not vanish. These elements lie in g_1 by the invariance. Lemma 4 then follows from Lemma 1′.

Now let V be a minimal invariant subspace of T_{x_0}. Thus $g|V$ is irreducible and we can apply the following:

LEMMA 5. *If* $g \subset \mathrm{Hom}(\mathbf{C}^n,\mathbf{C}^n)$ *is an irreducible Lie algebra of infinite type, it must be one of the following four algebras.*

(1) $sl(n,\mathbf{C})$.

(2) $gl(n,\mathbf{C})$.

(3) $sp(n,\mathbf{C})$.

(4) $sp(n,\mathbf{C}) + \{cI\}$.

(For the proof cf. ref. 2.)

From Lemma 4 it follows that $g|V$ must be one of the above algebras. Since $g_1|V$ is an ideal in $g|V$ and contains elements of rank one, there are only limited possibilities for it also, namely:

If $g|V$ is (1), $g_1|V$ is (1).

If $g|V$ is (2), $g_1|V$ is (1) or (2).
If $g|V$ is (3), $g_1|V$ is (3).
If $g|V$ is (4), $g_1|V$ is (3) or (4).

We will explore these various possibilities. First, we note that there is a natural mapping c: $V \wedge V \rightarrow W$. To define this, let v_1,v_2 be vectors in V and let X_1,X_2 be Γ vector fields such that $X_1 = v_1$ and $X_2 = v_2$ at x_0. If X_1' and X_2' are Γ vector fields having the same property at x_0, there exist $\alpha_i \epsilon g$, $u = 1,2$, such that $X_i - X_i' = \alpha_i$. Thus $[X_1,X_2]_{x_0} = [X_1',X_2']_{x_0} + \alpha_1(v_2) - \alpha_2(v_1)$. Since the last part of this expression is in V, the projection of $[X_1,X_2]_{x_0}$ on W is equal to the projection of $[X_1',X_2']_{x_0}$ on W. Thus we get a natural mapping of $V \wedge V$ into W, and this is c.

LEMMA 6. *c is equivariant under the induced representation of g on $V \wedge V$ and on W.*

The lemma is an immediate consequence of the definition of c and Jacobi's identity.

Since g_1 is zero on W, it follows from Lemma 6 that if g_1 is either 1,2, or 4, c must be zero. This is impossible unless $V = T_{x_0}$, since otherwise V would give rise to an invariant foliation. Thus, either $g_1 = 1, 2, 3,$ or 4 and $V = T_{x_0}$, or $g_1 = 3$. In the second case, the range of the mapping c: $V \wedge V \rightarrow W$ can be one-dimensional at most; otherwise, g_1 would have to leave invariant two linearly independent elements of $V \wedge V$, and this would exclude the symplectic algebra.

In the second case, let V' be the subspace of T_{x_0} spanned by V and the range of c. We will show this space must be all of T_{x_0}. It is g-invariant by Lemma 6; therefore, there is also a canonical mapping c': $V' \wedge V' \rightarrow W'$ associated with it. (We use the notation W' for T_{x_0}/V'.) From the definition, it follows that $c' = 0$ when restricted to $V \wedge V$. Let v' be any element of V' not in V, and let $c_{v'}$ be the mapping of V into W' defined by

$$c_{v'}(v) = c'(v' \wedge v).$$

This mapping is defined invariantly up to a scalar multiple, and its kernel, which is an invariant subspace, must be either V or $\{0\}$ because of the minimality of V. We will exclude the second possibility. If the kernel is $\{0\}$, $c_{v'}$ is injective. Taking $A \epsilon g_1$, we get

$$0 = Ac'(v'w) = c'(Av',w) + c'(v',Aw) = c_{v'}(Aw) \qquad \text{for } w \epsilon V,$$

thus, $Aw = 0$. We have shown, however, that $g_1|V$ is nonzero, in fact, of infinite type; so this case is ruled out.

If the kernel is V, then c' is identically zero on $V' \wedge V'$ (since $c'(v' \wedge v') = 0$). This means that V' corresponds to an integrable distribution, or, since Γ is primitive, $V' = T_{x_0}$.

Summarizing what we have proved so far:

THEOREM 1. *If Γ is an infinite primitive pseudogroup, either the linear isotropy algebra is irreducible and equal to one of the four algebras listed in Lemma 5, or there exists a one form, ω, on M such that ω is preserved up to a scalar multiple by Γ; and, restricted to the subspace $\omega = 0$ in T_{x_0}, the linear isotropy algebra is the symplectic algebra plus center: $sp(n,\mathbf{C}) + \{cI\}$.*

The only item in Theorem 1 we have not proved is that g, restricted to the subspace $\omega = 0$, is the algebra $sp(n,\mathbf{C}) + \{cI\}$. Our argument shows that it is either this or the algebra $sp(n,\mathbf{C})$. If it were $sp(n,\mathbf{C})$, Γ would actually have to leave the form ω invariant, and, therefore the one-dimensional foliation associated with $d\omega$; but this is excluded by the primitivity. q.e.d.

From Theorem 1 it is not difficult to determine Γ itself. For details we refer to reference 6. We state here the results:

If the linear isotropy algebra is (1), Γ is the pseudogroup of diffeomorphisms leaving a volume element on M fixed.

If the linear isotropy algebra is (2), Γ is either the pseudogroup of all diffeomorphisms of M, or the pseudogroup of diffeomorphisms which preserves a volume element up to a constant multiple.

If the linear isotropy algebra is (3), Γ is the pseudogroup of diffeomorphisms leaving fixed a symplectic structure on M.

If the linear isotropy algebra is (4), Γ is the pseudogroup of diffeomorphisms which preserves a 2-form of maximal rank up to a constant multiple.

If the linear isotropy algebra leaves invariant a hyperplane, Γ is the pseudogroup of diffeomorphisms leaving fixed a contact structure on M.

There are thus six distinct cases in all.

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*ON THE CONSTRUCTION OF DIVISION RINGS BY THE DEFORMATION OF FIELDS**

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This note presents certain explicit formulas for the deformation of an associative ring. Since a deformation of a division ring is again a division ring, while the formulas when applied to any ring generally produce a noncommutative ring, it follows that the formulas in particular give methods for constructing classes of division rings as deformations of fields. It is not known, in the finite-dimensional case, whether all the division rings so constructed are in fact crossed products; this is the principal open question raised here. The formulas given are special cases of ones which will be given in detail elsewhere.

1. Let A be an associative algebra over a field k. Following the author,[1] let a