this or the algebra $sp(n,\mathbf{C})$. If it were $sp(n,\mathbf{C})$, Γ would actually have to leave the form ω invariant, and, therefore the one-dimensional foliation associated with $d\omega$; but this is excluded by the primitivity. q.e.d.

From Theorem 1 it is not difficult to determine Γ itself. For details we refer to reference 6. We state here the results:

If the linear isotropy algebra is (1), Γ is the pseudogroup of diffeomorphisms leaving a volume element on M fixed.

If the linear isotropy algebra is (2), Γ is either the pseudogroup of all diffeomorphisms of M, or the pseudogroup of diffeomorphisms which preserves a volume element up to a constant multiple.

If the linear isotropy algebra is (3), Γ is the pseudogroup of diffeomorphisms leaving fixed a symplectic structure on M.

If the linear isotropy algebra is (4), Γ is the pseudogroup of diffeomorphisms which preserves a 2-form of maximal rank up to a constant multiple.

If the linear isotropy algebra leaves invariant a hyperplane, Γ is the pseudogroup of diffeomorphisms leaving fixed a contact structure on M.

There are thus six distinct cases in all.

* This work was supported by National Science Foundation grant GP-4433 and a grant from the Sloan Foundation.

[1] Cartan, E., "Les groupes de transformation continus, infinis, simples," *Ann. Sci. Ecole Norm. Sup.*, **26**, 93–161 (1909).

[2] Guillemin, V., D. Quillen, and S. Sternberg, "The classification of the irreducible complex algebras of infinite type," in preparation.

[3] Guillemin, V., and D. Quillen, "On Hamilton-Jacobi theory for over-determined systems," in preparation.

[4] Guillemin, V., and S. Sternberg, "Deformation theory of pseudogroup structures," submitted to *Memoirs Am. Math. Soc.*

[5] Ochiai, T., "Classification of the finite primitive Lie algebras," in preparation.

[6] Singer, I. M., and S. Sternberg, "On the infinite groups of Lie and Cartan, I," *J. Analyse Math.*, **15**, 1–114 (1965).

---

*ON THE CONSTRUCTION OF DIVISION RINGS BY THE DEFORMATION OF FIELDS**

BY MURRAY GERSTENHABER

UNIVERSITY OF PENNSYLVANIA AND THE INSTITUTE FOR ADVANCED STUDY

Communicated by A. A. Albert, February 17, 1966

This note presents certain explicit formulas for the deformation of an associative ring. Since a deformation of a division ring is again a division ring, while the formulas when applied to any ring generally produce a noncommutative ring, it follows that the formulas in particular give methods for constructing classes of division rings as deformations of fields. It is not known, in the finite-dimensional case, whether all the division rings so constructed are in fact crossed products; this is the principal open question raised here. The formulas given are special cases of ones which will be given in detail elsewhere.

1. Let A be an associative algebra over a field k. Following the author,[1] let a

multiplication π_t be introduced on the formal power series $\Sigma a_i t^i$ with coefficients in A by setting

$$\pi_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) + \ldots,$$

the F_i being bilinear functions $A \times A \rightarrow A$ which are extended to be bilinear over the ring of formal power series $k[[t]]$. Given bilinear functions $F, G : A \times A \rightarrow A$, if we define $F \circ G : A \times A \rightarrow A$ by setting

$$F \circ G\ (a, b, c) = F(G(a, b), c) - F(a, G(b, c)),$$

then the necessary and sufficient condition that π_t be an associative multiplication is that $\Sigma F_l \circ F_m = \delta F_{n'}$, where the sum is over all positive l and m such that $l + m = n$, and where δ denotes the Hochschild coboundary operator.[2]

It is trivial to demonstrate that if A has a unit, then so has the resulting algebra, and that if A is a division algebra and if we extend the coefficients from $k[[t]]$ to its quotient field $k((t))$, then the resulting algebra is again a division algebra. Note that if $F_n = 0$ for all sufficiently large n, then it is unnecessary to consider all power series; polynomials will do, and the deformed algebra is defined over $k(t)$.

Given linear functions $f, g:A \rightarrow A$, we define a bilinear function $f \cup g:A \times A \rightarrow A$ by setting $(f \cup g)(a, b) = f(a)\cdot g(b)$.

2. Let A be an algebra over a field of characteristic zero and φ, μ be commuting derivations of A. Then setting $F_n = (n!)^{-1}\varphi^n \cup \mu^n$ (where φ^n is just the nth iterate of φ), it is trivial to verify by direct computation that these F_n satisfy the required conditions and therefore give an explicit deformation of A. If there exist a and b in A such that $\varphi a\cdot\mu b \neq \varphi b\cdot\mu a$, then the deformed algebra is clearly noncommutative. This will generally be the case unless $\varphi = \mu$. But if A was a division ring, then so is the deformed algebra, which we shall denote by A_t; in particular, if A was a field, then A_t is a division ring. It is thus possible, for example, to deform the function field of an abelian variety of dimension at least two and characteristic zero into a noncommutative division ring, corresponding to the fact that the variety can itself be deformed into a nonalgebraic object. This was the original example discovered by Gherardelli and the author. Even simpler, the rational function field $k(x, y)$ in two variables can be deformed into a division ring over $k(t)$ by taking $\varphi = \partial/\partial x$, $\mu = \partial/\partial y$. The center of this ring is just $k(t)$. The examples of this section are all necessarily of infinite dimension.

3. Suppose now that A is an algebra over a field k of characteristic p with a pair of commuting derivations φ and μ such that $\varphi^p = \mu^p = 0$. Then letting π denote the original multiplication, the formal product

$$\pi_t = \pi + t\varphi\cup\mu + \frac{t^2}{2!}\,\varphi^2 \cup \mu^2 + \ldots + \frac{t^{p-1}}{(p-1)!}\,\varphi^{p-1} \cup \mu^{p-1}$$

is still associative. If A is a division ring, then the deformed algebra A_t is a division algebra over $k(t)$. The conditions for the construction are fulfilled, for example, when A is a purely inseparable field $k(x, y)$ of degree p^2 over k with x^p, y^p in k. We can then define $\varphi x = 1$, $\varphi y = 0$, $\mu x = 0$, $\mu y = 1$, and obtain, in fact, a central division algebra of characteristic p and dimension p^2 over $k(t)$. Note that both $k(t)(x)$ and $k(t)(y)$ are maximal subfields of A_t, and being simple, purely inseparable extensions of $k(t)$, it follows by a theorem of Albert[3] that A is in fact cyclic.

4. Suppose now that k is of characteristic two, that φ, μ are commuting derivations of A, and that linear functions φ', u': $A \to A$ are given with the properties: (1) $\delta\varphi' = \varphi \cup \varphi$ and $\delta\mu' = \mu \cup \mu$. This is equivalent to the assertion that $a \to a + t\varphi a + t^2\varphi' a$ is an automorphism of $A[t]/(t^3)$, and similarly for μ'; (2) φ, μ, φ', μ' all commute pairwise; (3) $\varphi^4 = \mu^4 = \varphi'^4 = \mu'^4 = 0$. Then the following may be verified by direct computation to be an explicit deformation of A:

$$\begin{aligned}\pi_t = \pi &+ t(\varphi \cup \psi) + t^2(\varphi' \cup \mu^2 + \varphi^2 \cup \mu') + t^3(\varphi\varphi' \cup \mu^3 + \varphi^3 \cup \mu\mu' + \varphi^3 \cup \mu^3) \\ &+ t^4(\varphi^2\varphi' \cup \mu^2\mu' + \varphi'^2 \cup \mu'^2) + t^5(\varphi^3\varphi' \cup \mu^3\mu' + \varphi\varphi'^2 \cup \mu\mu'^2) \\ &+ t^6(\varphi^2\varphi'^2 \cup \mu'^3 + \varphi'^3 \cup \mu^2\mu'^2) \\ &+ t^7(\varphi^3\varphi'^2 \cup \mu\mu'^3 + \varphi\varphi'^3 \cup \mu^3\mu'^2 + \varphi^3\varphi'^2 \cup \mu^3\mu'^2) \\ &+ t^8(\varphi^2\varphi'^3 \cup \mu^2\mu'^3) + t^9(\varphi^3\varphi'^3 \cup \mu^3\mu'^3).\end{aligned}$$

(The expression is less forbidding than it looks at first blush!)

The conditions are satisfied, for example, when A is a field of degree 2^8 over k of the form $A = k(x_1, x_2, x_3, x_4)$ with x_i^4 in k for all i, where one may define φ, μ, φ', μ' as follows: (1) $\varphi x_1 = x_2$, $\varphi x_2 = 1$, $\varphi x_3 = \varphi x_4 = 0$, $\mu x_1 = \mu x_2 = 0$, $\mu x_3 = x_4$, $\mu x_4 = 1$. (2) Define φ' and μ' first on kA^2 by setting $\varphi'(a^2) = (\varphi a)^2$, extended linearly—which is possible in this case—and similarly for μ'. Then set $\varphi'(1) = \varphi'(x_i) = \mu'(1) = \mu'(x_i) = 0$ for all i. (Given φ and μ, these are not unique.) The conditions $\delta\varphi' = \varphi \cup \varphi$ and $\delta\mu' = \mu \cup \mu$ then uniquely (and consistently) determine φ' and μ' on all of A. With the foregoing choices, A_t becomes a central division algebra of dimension 256 over $k(t)$ and contains $k(t)(x_1, x_2)$ and $k(t)(x_3, x_4)$ as maximal subfields. These are not simple extensions of $k(t)$, and it is presently not known whether the A_t here constructed is a cyclic algebra or not.

5. The preceding example suggests that in the case of characteristic p, if automorphisms Φ and Ψ of $A[t]/(t^{p^n+1})$ of the form $\Phi a = a + t\varphi_1 a + t^2\varphi_2 a + \ldots + t^{p^n}\varphi_{p^n} a$, and similarly for Ψ can be found with pairwise commuting φ_i and μ_j all nilpotent of index p^{n+1}, then an explicit deformation formula can be exhibited. Direct calculation in fact shows that this is the case. At present the only conceptual clue to this extraordinary phenomenon is the existence of nonalgebraic deformations of algebraic varieties in characteristic zero, as mentioned in §1.

* The author gratefully acknowledges the support of the National Science Foundation through grant GP-3683 to the University of Pennsylvania.

[1] Gerstenhaber, M., "On the deformation of rings and algebras," *Ann. Math.*, **79**, 59–103 (1964).

[2] Hochschild, G., "On the cohomology groups of an associative algebra," *Ann. Math.*, **46**, 58–67 (1945).

[3] Albert, A. A., *Structure of Algebras* (Providence: American Mathematical Society, 1961), Theorem 27, p. 107.