*HYPERFUNCTIONS AND LINEAR PARTIAL DIFFERENTIAL EQUATIONS**

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M. Sato's hyperfunctions are a generalization of distributions. Some applications are made to linear partial differential equations with constant coefficients. Global existence in the space of hyperfunctions holds for all open sets. Also, Weyl's lemma is extended to include hyperfunctions.

1. *Hyperfunctions of M. Sato.*[1]—Let Ω be an open set $\mathbf{R}^n$, and V an open set in $\mathbf{C}^n$ containing Ω as a relatively closed subset. $B(\Omega)$, the family of hyperfunctions on Ω, is by definition $H^n_\Omega(V, \mathcal{A})$, the nth relative cohomology group of V mod $V - \Omega$ with coefficients in the sheaf $\mathcal{A}$ of germs of analytic functions. In the notation of Godement,[2] $B(\Omega) = H^n_\Phi(V, \mathcal{A})$, where $\Phi = \{S: S \subset \Omega \text{ and } S \text{ is closed in } V\}$. It follows easily from the definition that $B(\Omega)$ is independent of the choice of V.

By a basic result of Grauert,[3] for each open set Ω in $\mathbf{R}^n$, we can find a fundamental neighborhood system in $\mathbf{C}^n$ consisting of domains of holomorphy. Therefore, we can pick a domain of holomorphy V so that $V \cap \mathbf{R}^n = \Omega$. In this case, $B(\Omega)$ may be identified with the Čech cohomology group $A(V \# \Omega)/\Sigma A(\hat{V}_j)$, where $A(V \# \Omega)$ and $A(\hat{V}_j)$ are the spaces of analytic functions on $V \# \Omega = (\mathbf{C} - \mathbf{R})^n \cap (V - \Omega)$ and on $\hat{V}_j = ((\mathbf{C} - \mathbf{R})^{j-1} \times \mathbf{C} \times (\mathbf{C} - \mathbf{R})^{n-j}) \cap (V - \Omega)$, respectively. If $u \in B(\Omega)$ is the equivalence class $[F]$ where $F \in A(V \# \Omega)$, we will call F a *defining function* for u.

The following basic results concerning hyperfunctions, (*a*) through (*e*), are due to Sato.[1] However, his complete proofs are not available. The brief development found here is based on Martineau's exposition.[4]

If K is a compact subset of $\mathbf{C}^n$, we denote by $A(K)$ the space of functions, analytic in some complex neighborhood of K, with the inductive-limit topology. Let $A'(K)$ denote the dual of $A(K)$. Elements of $A'(K)$ are called *analytic functionals.* If $K \subset \mathbf{R}^n$, Martineau has shown, using Serre duality, that $H^p_K(\mathbf{C}^n, \mathcal{A}) = 0$, for $p = 0, \ldots, n-1, n+1$, and that $H^n_K(\mathbf{C}^n, \mathcal{A})$ is isomorphic to $A'(K)$. These facts are enough to derive (*a*) through (*c*) below.

Let $\mathcal{B}$ denote the sheaf over $\mathbf{R}^n$ determined by the presheaf $\{B(\Omega): \Omega \text{ an open set in } \mathbf{R}^n\}$.

(a) *The presheaf* $B(\Omega)$ *is a sheaf, that is,* $B(\Omega) = \Gamma(\Omega, \mathcal{B})$.

Proof: As suggested by Sato,[1] once it is proved that $H^p_\Omega(V, \mathcal{A}) = 0$, $p = 0, \ldots, n-1$ for small Ω, then (*a*) follows by the method of spectral sequences. Consider the exact sequence of relative cohomology groups associated with the triple $\mathbf{C}^n \supset \mathbf{C}^n - \partial\Omega \supset \mathbf{C}^n - \bar{\Omega}$:

$$\ldots \rightarrow H^p_{\bar{\Omega}}(\mathbf{C}^n, \mathcal{A}) \rightarrow H^p_\Omega(\mathbf{C}^n - \partial\Omega, \mathcal{A}) \rightarrow H^{p+1}_{\partial\Omega}(\mathbf{C}^n, \mathcal{A}) \rightarrow \ldots.$$

Assume Ω is bounded. Then, since $\bar{\Omega}$ and $\partial\Omega$ are compact, it follows that $H^p_\Omega(\mathbf{C}^n - \partial\Omega, \mathcal{A}) = 0$ for $p = 0, \ldots, n-2$. The natural map of $H^n_{\partial\Omega}(\mathbf{C}^n, \mathcal{A})$ into $H^n_{\bar{\Omega}}(\mathbf{C}^n, \mathcal{A})$ is injective since the map of $A(\bar{\Omega})$ into $A(\partial\Omega)$ has dense range. Therefore $H^{n-1}_\Omega(\mathbf{C}^n - \partial\Omega, \mathcal{A}) = 0$.

Because of (a), we have a concept of support for hyperfunctions. Let κ be a relatively closed subset of the open set $\Omega \subset \mathbf{R}^n$. Let V be an open set in $\mathbf{C}^n$ containing Ω as a relatively closed subset. Denote by $B_\kappa(\Omega)$ the space of sections, of the sheaf $\mathcal{B}$ on Ω, with support in κ. Then

(b) $B_\kappa(\Omega) = H^n_\kappa(V, \mathcal{A})$. *In particular, if K is a compact subset of* $\mathbf{R}^n$, $B_K(\mathbf{R}^n) \cong A'(K)$.

Proof: $B_\kappa(\Omega)$ is by definition the kernel of the restriction map of $B(\Omega)$ into $B(\Omega - \kappa)$. On the other hand, the triple $V \supset V - \kappa \supset V - \Omega$ gives the exact sequence:

$$0 = H^{n-1}_{\Omega-\kappa}(V - \kappa, \mathcal{A}) \to H^n_\kappa(V, \mathcal{A}) \to H^n_\Omega(V, \mathcal{A}) \to H^n_{\Omega-\kappa}(V - \kappa, \mathcal{A}).$$

(c) *B is flabby.*

Proof: Since $H^{n+1}_{\partial\Omega}(\mathbf{C}^n, \mathcal{A}) = 0$, the exact sequence

$$H^n_{\bar{\Omega}}(\mathbf{C}^n, \mathcal{A}) \to H^n_\Omega(\mathbf{C}^n - \partial\Omega, \mathcal{A}) \to H^{n+1}_{\partial\Omega}(\mathbf{C}^n, \mathcal{A})$$

shows that the restriction map $B_{\bar{\Omega}}(\mathbf{R}^n) \to B(\Omega)$ is onto.

Since $\mathcal{B}$ is flabby and $B_K(\mathbf{R}^n) \cong A'(K)$, every hyperfunction can be expressed as a locally finite sum of analytic functionals.[2] An easy consequence of this is that every distribution is a hyperfunction.

Let $A(\Omega)$ denote the real analytic functions on Ω. If $u \in A(\Omega)$, then $u \in A(V)$ for some complex Stein neighborhood V of Ω. Let χ^+ denote the characteristic function of $\{z : z \in V, \operatorname{Im} z_j > 0, j = 1, \ldots, n\}$.

(d) *If u is a real analytic function on Ω, then $\chi^+ u \in A(V \# \Omega)$ is a defining function for u on Ω.*

Proof: Since being a defining function is a local property, it is enough to pick $\varphi \in C_0^\infty(\Omega)$, such that φ is identically one on $\omega \subset\subset \Omega$, and show that $\chi^+ u$ is a defining function for φu on ω. Given $v \in A'(K)$, where K is a compact subset of $\mathbf{R}^n$, $\breve{v}(z) = (2\pi i)^{-n} v_t(\Pi(t_j - z_j)^{-1})$ is analytic on $(\mathbf{C} - \mathbf{R})^n$ and is a defining function for v on $\mathbf{R}^n$. In our case, $\breve{\varphi u}$ is a defining function for u on ω. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ where each $\sigma_j = \pm 1$ and let $V^\sigma = \{z : z \in V, \sigma_j \operatorname{Im} z_j > 0, j = 1, \ldots, n\}$.

$$\breve{\varphi u}\big|_{V^\sigma}(z) = (2\pi i)^{-n} \int_{\mathbf{R}^n} \frac{\varphi(t) u(t) dt}{\Pi(t_j - z_j)}, \qquad z \in V^\sigma,$$

can be analytically continued across ω by shifting the path of integration, since $\varphi u \in A(\omega)$. Let F^σ denote this prolonged function. Then by Cauchy's integral formula one can easily check that

$$u(z) = \Sigma(\operatorname{sgn} \sigma) F^\sigma(z) \tag{1}$$

for all z in some complex neighborhood W of ω. Let χ^σ denote the characteristic function of V^σ. From (1) one can show that $\chi^+ u - \Sigma \chi^\sigma F^\sigma \in \Sigma A(\hat{W}_j)$. But $\breve{\varphi u} = \Sigma \chi^\sigma F^\sigma$, completing the proof.

The next proposition follows immediately from (d) and its proof.

(e) *If $F \in A(V \# \Omega)$ is a defining function for $u \in B(\Omega)$ such that each $F|_{V^\sigma}$ can be continued across Ω to a function F^σ, then u is the real analytic function $\Sigma(\operatorname{sgn} \sigma)F^\sigma$.*

2. *Linear Partial Differential Equations with Constant Coefficients.*—If $u = [F] \in B(\Omega)$, then by definition $(\partial/\partial x_j)u = [(\partial/\partial z_j)F]$. The hyperfunctions on Ω obviously form a module over $A(\Omega)$. Hence, we can consider linear partial differ-

ential operators $P(x,D)$ with coefficients in $A(\Omega)$ acting on $B(\Omega)$ in the obvious way.

Since $\mathcal{B}$ is flabby, we have

LEMMA 1. *If* $\Omega \subset \Omega'$ *are two open sets in* $\mathbf{R}^n$ *and* $P(x,D)B(\Omega') = B(\Omega')$, *then* $P(x,D)\, B(\Omega) = B(\Omega)$.

The following lemma is often used to prove existence or approximation.[6] Let E and E_1 be reflexive Frechet spaces, and π a continuous linear map of E into E_1 with dense range. Then E_1' can be considered a subspace of E'. Let T and S be continuous linear maps of E into E, and E_1 into E_1, respectively, such that $\pi T = S\pi$.

LEMMA 2. *Suppose T is onto. Then S is onto and* $\pi[\ker T]$ *is dense in* $\ker S$ *if and only if it follows from* $u \in E'$ *and* $T^*u \in E_1'$ *that* $u \in E_1'$.

Consider the case where $E = A(\mathbf{C}^n)$, $F = A(W)$, and $T = P(D) = S$. Malgrange[6] has a short proof that $P(D)\, A(\mathbf{C}^n) = A(\mathbf{C}^n)$ and that the exponential polynomial solutions of $P(D)u = 0$ are dense in $N = \{u{:}u \in A(\mathbf{C}^n) \text{ and } P(D)u = 0\}$. Hence we have, as a special case, that: $P(D)A(W) = A(W)$ and the exponential polynomial solutions of $P(D)u = 0$ are dense in $N = \{u{:}u \in A(W) \text{ and } P(D)u = 0\}$ if and only if $v \in A'(\mathbf{C}^n)$ and $P^*(D)\, v \in A'(W)$ imply $v \in A'(W)$.

LEMMA 3. *For all open convex sets W in* $\mathbf{C}^n$, $P(D)A(W) = A(W)$, *and the exponential polynomial solutions of* $P(D)u = 0$ *are dense in* $\{u{:}u \in A(W) \text{ and } P(D)u = 0\}$.

Proof: If F is entire, let $M(z, F, r) = \sup\limits_{|z - \zeta| < r} |F(\zeta)|$. Suppose $v \in A'(\mathbf{C}^n)$ and $P(D)\, v \in A'(W)$. From Hörmander,[7] we know $|\hat{v}(z)| \leq [M(z, P\hat{v}, 4r)\, M(z, P, 4r)]/M(z, P, r)^2$, $z \in \mathbf{C}^n$, and, since P is a polynomial, that there exists a constant $\eta > 0$ independent of z such that $M(z, P, r) > \eta$, $z \in \mathbf{C}^n$. Combining these two facts we can conclude that $v \in A'(W)$ by the Polya-Ehrenpreis-Martineau theorem.[5]

From now on, $P(D)$ will denote both the operator $\Sigma a_\alpha(-i\partial/\partial x)^\alpha$ on $B(\Omega)$ and the operator $\Sigma a_\alpha(-i\partial/\partial z)^\alpha$ on $A(W)$.

THEOREM 1. *For each open set* Ω *in* $\mathbf{R}^n$, $P(D)\, B(\Omega) = B(\Omega)$.

Proof: By Lemma 1 we need only show that $P(D)\, B(\mathbf{R}^n) = B(\mathbf{R}^n)$. However, since each of the 2^n connected components of $\mathbf{C}^n \# \mathbf{R}^n$ is convex, $P(D)\, A(\mathbf{C}^n \# \mathbf{R}^n) = A(\mathbf{C}^n \# \mathbf{R}^n)$ by Lemma 3. Apply this to the defining functions.

Of course, one gets entirely different results if $D'(\Omega)$ or $\mathbf{C}^\infty(\Omega)$ are substituted for $B(\Omega)$ in Theorem 1 (see, for example, Hörmander[7]).

THEOREM 2. *The following are equivalent:*[8]

(a) $P(D)$ *is elliptic.*

(b) *If* $u \in B(\Omega)$ *and* $P(D)u \in A(\Omega)$, *then* $u \in A(\Omega)$.

(c) *If* $u \in B(\Omega)$ *and* $P(D)u \in C^\infty(\Omega)$, *then* $u \in C^\infty(\Omega)$.

Proof that (*a*) *implies* (*b*)*:* Since real analyticity is a local property, it is enough to prove (*b*) in the case $\Omega = \{x{:}|x_j| < r, j = 1,\ldots, n\}$. Also, by the Cauchy-Kovalevsky theorem we can locally solve $P(D)v = f$ with v analytic so we may assume $f = 0$. Let $V = \{z{:}|z_j| < r, j = 1,\ldots, n\}$. Let $G \in A(V \# \Omega)$ be a defining function for u on Ω. Then $P(D)u = 0$ means $P(D)G = \Sigma H_j$, where $H_j \in A(\hat{V}_j)$. By Lemma 3 there exist $G_j \in A(\hat{V}_j)$ such that $P(D)\, G_j = H_j$. Then $F = G - \Sigma G_j$ is a defining function for u and $P(D)F = 0$. By (*e*) of § 1, this reduces the proof to the following lemma which is concerned with natural domains of existence for certain overdetermined systems.

LEMMA 3. *Suppose $P(D)$ is elliptic. Then for each neighborhood V of 0 in $\mathbf{C}^n$, there exists a neighborhood U of 0 in $\mathbf{C}^n$ such that, for every component V^σ of $V \# (V \cap \mathbf{R}^n)$, if $F \in A(V^\sigma)$ and $P(D)F = 0$, then F can be analytically continued to all of U.*[9]

Sketch of proof: Let $N(W) = \{F:F \in A(W) \text{ and } P(D)F = 0\}$. Let $V_\rho = \{z: |\mathrm{Im}\, z_j| + |\mathrm{Re}\, z_j| < \rho\}$. We will show that for each $\rho > 0$ there exists a $\delta > 0$ such that the restriction map $r:N$ (convex hull of $V_\rho^\sigma \cup U_\delta) \rightarrow N(V_\rho^\sigma)$ is onto, where, in our calculations below, we will take U_δ to be the product of the n triangles T_δ^j with vertices $(0, -\delta)$, (δ, δ), $(-\delta, \delta)$. Let $H_K(z) = \sup_{\zeta \in K} \mathrm{Re}\, z\zeta$. By the Polya-Ehrenpreis-Martineau theorem, for all convex compact sets K in $\mathbf{C}^n$, $u \in A'(K)$ if and only if for each $\epsilon > 0$ there exists a constant C_ϵ such that $|\hat{u}(z)| \leq C_\epsilon \exp [H_K(z) + \epsilon|z|]$, $z \in \mathbf{C}^n$. An examination of the analytic uniform structure[10] $\psi(W) = \{a(z):a(z)$ is a continuous positive function on $\mathbf{C}^n$, $\exp [H_K(z)] = o(a(z))$ for all compact convex subsets K of $W\}$ on the open convex set W, and the comparison theorem of Ehrenpreis[10] show that (*i*) below is sufficient to prove the lemma. We will just examine the case $\sigma = (1,\ldots, 1)$ and let $V_\rho^+ = V_\rho^\sigma$. Let K_ϵ denote the product of the n triangles K_ϵ^j with vertices $(\rho - 2\epsilon, \epsilon)$, $(0, \rho - \epsilon)$, $(-(\rho - 2\epsilon), \epsilon)$. Let $K_{\epsilon,\delta}$ denote the convex hull of $K_\epsilon \cup \bar{U}_\delta$.

(i) *For all $\rho > 0$ there exist a $\delta > 0$ and a constant $c > 0$ such that for ϵ sufficiently small, $P(z) = 0$ implies $H_{K_{\epsilon,\delta}}(z) \leq H_{K_\epsilon}(z) + C$.* The maximum of $\mathrm{Re}\, z\zeta$ over K must occur at an extreme point since $\mathrm{Re}\, z\zeta$ is linear in z, and K is convex and compact. Therefore, $H_{K_{\epsilon,\delta}}(z) = \max \{H_{K_\epsilon}(z), H_{\bar{U}_\delta}(z)\}$. Hence, we may replace $H_{K_{\epsilon,\delta}}$ by $H_{\bar{U}_\delta}$ in (*i*). Now, one can calculate $H_{\bar{U}_\delta}$ and H_{K_ϵ} explicitly. Since $P(D)$ is elliptic, there exist constants M_1 and M_2 such that $P(z) = 0$ implies $\Sigma|\mathrm{Im}\, z_j| \leq M_1[1 + \Sigma|\mathrm{Re}\, z_j|]$ and $\Sigma|\mathrm{Re}\, z_j| \leq M_2[1 + \Sigma|\mathrm{Im}\, z_j|]$. Using these estimates, the proof of (*i*) is easy since $H_{\bar{U}_\delta}$ and H_{K_ϵ} are linear functions of $\Sigma|\mathrm{Re}\, z_j|$ and $\Sigma|\mathrm{Im}\, z_j|$.

Since we can locally solve $P(D)u = f$ for $u \in C^\infty$ if $f \in C^\infty$, (*b*) trivially implies (*c*). The fact that (*c*) implies (*a*) is a consequence of the following lemma. However, we can give an easy indirect proof that (*c*) implies (*a*). Since Theorem 1 holds with $B(\Omega)$ replaced by $C^\infty(\Omega)$ only if $P(D)$ is elliptic,[7] we know that we can find Ω, $f \in C^\infty(\Omega)$, and $u \in B(\Omega)$ but $u \notin C^\infty(\Omega)$ such that $P(D)u = f$. Our theorem says that the concepts of hypoellipticity and ellipticity are equivalent with respect to hyperfunctions.

LEMMA 4. *If $P(D)$ is not elliptic, then there exists a hyperfunction $u \in B(\mathbf{R}^n)$ such that $P(D)u = 0$ but $u \notin C^\infty(\mathbf{R}^n)$.*

Proof: Since $P(D)$ is not elliptic, there exists a real vector $N \neq 0$ such that $P_m(N) = 0$. By a linear change of coordinates we may assume $N = (1, 0,\ldots, 0)$. Pick a real vector M perpendicular to N so that $P_m(M) \neq 0$. Now consider $P(sN + tM) = 0$ as an equation in t with s fixed large. We can solve for t as a function of s and expand in the Puiseux series[11] $t(s) = s\sum_1^\infty c_j(s^{-1/p})^j$, for some integer p and $|s^{1/p}| > C$. Thus $|t(s)| \leq C'|s|^{1-(1/p)}$ if $|s| > (2C)^p$. Consider $F(z_1, z') = \frac{1}{2\pi}\int_0^\infty \exp [i(sz_1 + t(s)(M,z'))]\, ds$, $\mathrm{Im}\, z_1 > 0$, $z' \in \mathbf{C}^{n-1}$. Thus $P(D)F = 0$ whenever this integral is convergent. If $\mathrm{Im}\, z_1 > 0$ and s is sufficiently large, then

$$\begin{aligned}\text{Im}\,(sz_1 + t(s)(M,z')) &\geq s\,\text{Im}\,z_1 - |t(s)|\,|(M,z')| \\ &\geq s\,\text{Im}\,z_1 - C's^{1-(1/p)}|(M,z')| \geq \alpha s\end{aligned}$$

for some $\alpha > 0$. Therefore the integral converges and $F(z,z')$ is an analytic function of z, for $\text{Im}\,z_1 > 0$. Also $F(z_1,0) = -1/(2\pi_i z_1)$. Let χ denote the characteristic function of $\{z:\text{Im}\,z_j > 0\}$. Let $u = [\chi F]$. Then $u \in B(\mathbf{R}^n)$ and $P(D)u = 0$. Also, u is real analytic in $x_2,\ldots,x_n$, but $u(x_1, 0,\ldots, 0) \not\in L_l^{\text{loc}}(R)$.

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[8] Professor Bengel has independently proved that (*a*) and (*b*) are equivalent (personal communication).

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*THE PLEMELJ FORMULAS WITH UNRESTRICTED APPROACH, AND THE CONTINUITY OF CAUCHY-TYPE INTEGRALS**

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Let J be a Jordan arc or a Jordan curve, and suppose that J is oriented and rectifiable. Assume that the function $f(\zeta)$ is defined and summable on J. Then the Cauchy-type integral

$$\int_J \frac{f(\zeta)d\zeta}{\zeta - z}, \qquad (z \not\in J) \tag{1}$$

represents a holomorphic function in each region complementary to J. If $\zeta_0 \in J$ but ζ_0 is not an end point of J, then the singular Cauchy integral

$$\int_J \frac{f(\zeta)\,d\zeta}{\zeta - \zeta_0}, \qquad (\zeta_0 \in J) \tag{2}$$