THE RESIDUE CALCULUS AND SOME TRANSCENDENTAL RESULTS IN ALGEBRAIC GEOMETRY, I*

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The purpose of this paper and the following one[1] is to outline certain developments in the analytical theory of algebraic manifolds which seem to have applications to moduli and other questions.

Outline of the Problem.—The general question is to determine the behavior of the periods of the integrals on an algebraic family $\{V_t\}_{t\epsilon B}$ of algebraic manifolds.

An algebraic family of algebraic manifolds shall mean that we are given a nonsingular algebraic manifold W, a variety B, and a subvariety $\mathcal{V} \subset W \times B$ such that the $V_t = \mathcal{V} \cdot W \times \{t\}$ are hypersurfaces. Generally, the V_t will have at most ordinary singularities for t in $B - S$, S being a proper subvariety of B. Also, there may or may not be a base locus, depending on the problem at hand.

Now the periods of the integrals will have to mean the periods of all the $(q-r,r)$ forms on V_t; that is, the position of $H^{q-r,r}(V_t)$ as a direct summand of $H^q(V)$, where V is thought of as a fixed topological model of the V_t for t lying off S. It will *not* suffice to consider only the abelian integrals, that is, $H^{q,0}(V_t)$, nor will it suffice to consider the differentials of the second kind. For curves, the integrals of the second kind fill up $H^1(V_t)$; for surfaces, the deficiency is the Picard number ρ. For $q > 2$, the integrals of the second kind fill only a small part of $H^q(V_t)$.

In fact, for $q \geqq 2$, there is the additional difficulty that a differential of the second kind may be exact and still have periods. For $q = 2$, these periods may be assumed constant since a second kind of 2 form has no residues, but even this fails for $q > 2$.

On the face of it, the periods of the $(q-r,r)$ forms are not holomorphic functions of t directly. However, the mapping $\Phi: B \rightarrow$ "period variety of V" will be analytic, provided we interpret "period variety" as follows: Let $\mathfrak{F}_q$ be the flag manifold of increasing sequences of all subspaces $S_1 \subset S_2 \ldots S_q \subset H^q(V)$ where $\dim S_k = h^{q,0} + h^{q-1,1} + \ldots + h^{q-k+1,k-1}$. If we then map B into $\mathfrak{F}_q$ by $\Phi: B \rightarrow \mathfrak{F}_q$, where

$$\Phi(t) = H_t^{q,0} \subset H_t^{q,0} + H_t^{q-1,1} \subset \ldots \subset H_t^{q,0} + \ldots + H_t^{1,q-1} \subset H^q(V),$$

then Φ will be holomorphic. The precise statement is:

Theorem. (a) *Let $\rho_t: T_t(B) \rightarrow H^1(V_t, \theta_t)$ be the Kodaira-Spencer mapping and assume $\rho_t\, \partial/\partial t^\alpha = 0$; $t^1, \ldots, t^m$ being local coordinates on B. Then Φ is holomorphic.*

(b) *The adjoint mapping $\Phi^*: T_{\Phi(t)}(\mathfrak{F}_q)^* \rightarrow T_t(B)^*$ to the differential of Φ satisfies the following commutative diagram:*

$$\begin{array}{ccc} & \Sigma\, H_t^{q-r,r} \otimes H_t^{r+1,n-r-1} & \\ \swarrow{\scriptstyle \Phi^*} & & \searrow{\scriptstyle \mu} \\ T_t(B)^* & \xleftarrow{\ \rho_t^*\ } & H^{n-1}(V_t, \Omega_t^1 \otimes \Omega_t^n) \end{array} \qquad \text{(C)}$$

where μ is the cup product in cohomology.

We may see that Φ agrees with what we should mean by the period mapping, since the minors in the period matrix of V_t give the Plücker coordinates of the point

$\Phi(t)$ in $\mathfrak{F}_q$. The above explanations noted, we may state our central problem: Study the singularities of the mapping $\Phi: B \to \mathfrak{F}_q$.

1. *Representation of Cohomology by Residues.*—In practice, it is desirable to have the nonconstant periods of the $(q-r,r)$ forms represented explicitly by integrals of rational differentials. This can be done, at least locally for t in a polycylinder Δ in B, by means of the residue calculus in several complex variables. The idea is to represent $\{V_t\}_{t\epsilon\Delta}$ as a family of positive hypersurfaces in an algebraic manifold W, and then represent the variable periods on V_t as integrals of holomorphic forms in $W - V_t$. In doing this we must allow V_t to have ordinary singularities.

The essential case is the following situation: W is a closed algebraic manifold, $L \to W$ is a positive line bundle, $\{\sigma_t\}$ is a pencil of holomorphic sections of L, and V_t is given by $\sigma_t(w) = 0$.

If now φ is a rational $q + 1$ form on W with values in L^{r+1}, i.e.,

$$\varphi \epsilon H^\circ(W, \Omega_W{}^{q+1}(L^{r+1})),$$

then $\omega_t = \dfrac{\varphi}{(\sigma_t)^{r+1}}$ is a rational $q + 1$ form with poles on V_t, and so, if ω_t is closed, it defines a class in $H^{q+1}(W - V_t)$. In case V_t is nonsingular, ω_t defines a class $i(\omega_t)$ in $H^q(V_t)$ by *integration over the fiber:* if $\tau: H_q(V_t) \to H_{q+1}(W - V_t)$ is the mapping which sends a q cycle γ on V_t into the boundary of the tube lying over γ, then $i(\omega_t)$ is given by

$$\langle i(\omega_t), \gamma\rangle = \int_{\tau(\gamma)} \omega_t. \tag{I}$$

If V_t is irreducible with ordinary singularities and dim $V_t \leqq 3$, then we modify as follows: Letting $\tilde{V}_t \xrightarrow{\pi} V_t$ be the (canonical) desingularization, then the *variable* periods on $\tilde{V}_t$ all occur on cycles $\gamma \epsilon H_q(\tilde{V}_t)$ which may be pulled off the singular locus, and so $\tau(\pi(\gamma))$ is defined and contributes to $H_{q+1}(W - V_t)$. However, $\tau(\pi(\gamma))$ does *not* depend only on the homology class of $\pi(\gamma)$ in $H_q(\tilde{V}_t)$, and so we should consider the subgroup $\tilde{H}^{q+1}(W - V_t) \subset H^{q+1}(W - V_t)$ of classes ω_t orthogonal to certain inessential cycles in $H_{q+1}(W - V_t)$. The forms in $\tilde{H}^{q+1}(W - V_t)$ are precisely those forms $\omega_t = \dfrac{\varphi(w, t)}{(\sigma_t)^{r+1}}$ where $\varphi(w, t)$ satisfies adjoint conditions.

The final result is given by:

THEOREM. *Assume* V_t *is nonsingular or else* dim $V_t \leqq 3$.

(a) $H^q(\tilde{V}_t)$ *is given by* $\{H^q(W) \cap H^q(\tilde{V}_t)\} \oplus i\{\tilde{H}^{q+1}(W - V_t)\}$ $(q \leqq \dim V_t)$.

(b) *Furthermore,* $H^{q-r,r}(\tilde{V}_t)$ *is given by* $H^{q-r,r}(W) \cap H^q(\tilde{V}_t)$ *plus the classes* $\omega_t = \dfrac{\varphi}{(\sigma_t)^{r+1}}$, *reduced modulo the forms* $\dfrac{\psi}{(\sigma_t)^k}$, $k < r + 1$, *where* ψ *is a holomorphic* L^k*-valued form on* W.

This theorem explains the above flag construction plus the fact that the periods, so interpreted, are analytic functions of t.

As an example, we write down the cohomology of a quartic in P_3. Let then $W = P_3$ with affine coordinates $x,y,z;$ V be the surface given by $F(x,y,z) = x^4 + y^4 + z^4 - 1 = 0$, and $H^{1,1}(V)_0$ the primitive classes. Then $H^{2,0}(V)_0$ is generated by $\dfrac{dxdydz;}{F}$

$H^{1,1}(V)_0$ by the 19 classes $\dfrac{x^\alpha y^\beta z^\gamma dxdydz}{F^2}$ $(2 \leqq \alpha + \beta + \gamma \leqq 4;\ \alpha,\beta,\gamma \leqq 2)$; and

$H^{0,2}(V)$ by $\dfrac{x^2y^2z^2dxdydz}{F^3}$.

2. *The Differential Equation of the Periods.*—In this section we shall assume that V_t is generally nonsingular, although all the results go through for V_t with ordinary singularities and dim $V_t \leqq 2$.

Let $\omega = \dfrac{\varphi}{(\sigma_t)^{r+1}}$ as in §2 above. Then

$$\frac{\partial\omega}{\partial t} = \omega^{(1)} = \frac{\varphi_1}{(\sigma_t)^{r+2}}, \ldots, \frac{\partial^k\omega}{\partial t^k} = \omega^{(k)} = \frac{\varphi_k}{(\sigma_t)^{r+k+1}}.$$

Since dim $H^{q+1}(W - V_t) < \infty$, after finitely many differentiations there is a linear relation

$$\omega^{(m)} + r_1(t)\omega^{(m-1)} + \ldots + r_m(t)\omega = d\xi(t).$$

Furthermore, the $r_\alpha(t)$ have only poles, and so are rational functions of t. If now $\gamma \epsilon H_q(V_t)$, $\pi_\gamma(t) = \displaystyle\int_{\tau(\gamma)} \omega$, then $\pi_\gamma(t)$ satisfies a linear differential equation:

$$\pi_\gamma^{(m)}(t) + r_1(t)\pi_\gamma^{(m-1)}(t) + \ldots + r_m(t)\pi_\gamma(t) = 0. \qquad \text{(E)}$$

A fundamental result is:

THEOREM. (E) *has only regular singularities in the sense of Fuchs. Thus the entries in the period matrix* $\Phi(t)$ *are locally solutions of a Fuchsian differential equation.*

This then sheds light on the singularities of the mapping Φ. If t_0 is a critical point of (E), then $|(t-t_0)^\lambda \pi_\gamma(t)| < \infty$ for some λ, so that the singularities of (E) are, in this sense, not essential.

It is interesting that the equation (E) is frequently not difficult to compute. We give some examples.

(*i*) $W = P_2$, $\sigma_t = y^2 - x^3 + t$, $\omega = \dfrac{dxdy}{\sigma}$ (pencil of elliptic curves acquiring a cusp). Then $\omega' - \dfrac{1}{6t}\omega = d\xi$, so $\pi'(t) - \dfrac{1}{6t}\pi(t) = 0$.

The reason this equation is first order is that the V_t are birationally equivalent for $t \neq 0$.

(*ii*) $W = P_{n+1}$, $\sigma_t = x_1^2 + \ldots + x_{n+1}^2 - t$ (pencil of quadrics acquiring an isolated doublepoint), $\omega = \dfrac{dx_1 \ldots . dx_n}{\sigma}$. Then $\omega' - \dfrac{(n-1)}{2t}\omega = d\psi$.

(*iii*) $W = P_3$, $\sigma = z^2 + y^2 - t(x^2 - 1)$, $\omega = \dfrac{dxdydz}{\sigma}$. Then

$$\omega'' + \frac{3}{4t}\omega' - \frac{3}{8t^2}\omega = d\psi.$$

(*iv*) $W = P_2$, $\sigma = y^2 - x(x-1)(x-t)$, $\omega = \dfrac{dxdy}{\sigma}$.

Then

$$\omega + \frac{(1 - 2t)}{t(1 - t)} \omega' - \frac{1}{4t(1 - t)} \omega = 0.$$

3. *Singularities of the Differential Equation.*—The analysis of the singularities of Φ depends therefore on the singularities of the equation (E), and this in turn has a close relation with the topological nature of the singularities of the special V_t. More precisely, we restrict our attention to the t-disk Δ and assume that V_t are homeomorphic and have ordinary singularities for $t \neq 0$.

We may choose a local basis $\pi_1, \ldots, \pi_m$ of the solutions to (E) of the form

$$\pi_j = \int_{\tau(\gamma_j)} \omega,$$

where $\gamma_1, \ldots, \gamma_m$ are q-cycles in $H_q(V_t)$. As t turns around zero, analytic continuation of $\pi_j(t)$ leads to a substitution

$$\pi_j \rightarrow \sum_{k=1}^{m} a_{kj} \pi_k.$$

In fact, it is clear that $A = (a_{kj})$ is the matrix of the *monodromy transformation*

$$T(\gamma_j) = \sum_{k=1}^{m} a_{kj} \gamma_k \qquad \text{(S)}$$

obtained by translating the cycles $\gamma_1, \ldots, \gamma_m$ around $t = 0$.

THEOREM. *The matrix A has integer entries, and all eigenvalues are roots of unity.*

The conclusion is that, in the first place, (E) is a special Fuchs' equation; and secondly, from the theory of regular differential equations, the singularities of $\pi_1, \ldots, \pi_m$ depend on knowing precisely the substitution (S). For example, if $\pi = \Sigma \lambda_j \pi_j$ is an eigenfunction so that $\pi \rightarrow \xi \pi$ with $\xi = e^{\frac{2\pi i p}{q}}$, then $\pi(t) = t^{p/q} \eta(t)$, where $\eta(t)$ is single-valued and has a finite pole at $t = 0$.

The determination of (S) is a topological question, generally quite complicated, which will be discussed in the next section. However, it is perhaps worth while to note that, in some simple cases, the form of (S) can be deduced from the equation (E). In fact, if we know (E), then we may deduce the Jordan normal form of A by a standard procedure, and it may be possible to recover the integral transformation (S). Referring to examples (*i*), (*ii*), and (*iv*) of §2, we list the corresponding results.

(*i*) Here $m = 2$ and A is an integer matrix of order 6; it is easy to check that A is equivalent to $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

(*ii*) Here $m = 1$ and $A = (-1)^{n-1}$.

(*iii*) Around zero, $m = 2$ and $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. The two solutions here are $\pi_1(t)$ = analytic function of t, $\pi_2(t) = \frac{1}{\pi \sqrt{-1}} \pi_1(t) \log t$.

4. *Topological Determination of the Monodromy Substitution.*—Let $\{V_t\}_{t \epsilon \Delta}$ be a fiber space of algebraic manifolds fibered over the disk Δ. For $t \neq 0$, we assume that V_t is locally given by

$$z_1 \ldots z_k = 0,$$

where $z_1, \ldots, z_{n+1}$ are coordinates in $W = V_t$; and we assume that the V_t are homeomorphic. As t turns around zero, there is induced an automorphism

$$T: H_q(V) \rightarrow H_q(V) \qquad (V = V_{t_0} \text{ for fixed } t_0 \neq 0).$$

On the other hand, we may let $E_q \subset H_q(V)$ be the subspace of cycles γ which vanish in V_0 (these are the *vanishing cycles*) and the main general result is:

THEOREM. *For* $\gamma \in H_q(V)$, $\gamma - T(\gamma) \in E_q$.

Thus $T(E_q) \subset E_q$ and, if $\delta_1, \ldots, \delta_m$ generate E_q,

$$T(\gamma) = \gamma + \sum_{j=1}^{m} \lambda_j(\gamma)\delta_j,$$

where the λ_j are linear forms on $H_q(V)$. Two applications of the above result are:

(α) If $q = n$ and if $|(\delta_1, \delta_j)| \neq 0$, (,) being the intersection pairing, then $H_q(V) = E_q \oplus F_q$ and $T = T_{E_q} \oplus I$.

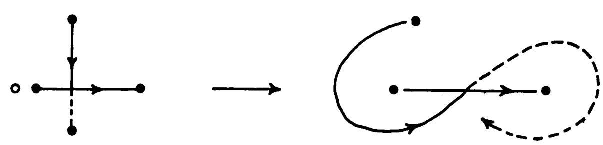
The advantage of this fact is that the determination of T_{E_q} is a local problem.

(β) If T_{E_q} is of finite order, then a basis to the solutions of (E) is of the form $t^\alpha\{\varphi(t) + \psi(t) \log t\}$, where φ, ψ are holomorphic and α is rational. (If T_{E_q} is infinite order, then we find terms $t^\alpha(\log t)^k$ $(k > 1)$.)

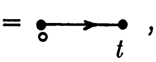
Finding T in a particular case is usually an interesting problem. The following four steps seem to give a good method of determining T in many cases; they are especially effective when V_0 is locally a union of hypersurfaces.

(1) Determination of T when dim $V = 1$. This can usually be done by drawing a suitable picture.

Examples: (e_1) Locally, $\{V_t\}$ is given by $y^2 = x(x - t)$. Then the picture

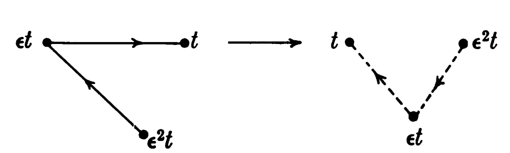


shows that $\begin{cases} \delta \rightarrow \delta \\ \gamma \rightarrow \gamma + 2\delta, \end{cases}$ where

$\delta =$ , $\gamma =$  (δ generates E_1).

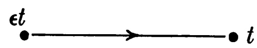
Thus $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, as has been checked using the differential equation.

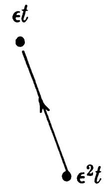
(e_2) Locally, $\{V_t\}$ is given by $y^2 = x^3 - t$. Then the picture



$(\epsilon = e^{\frac{2\pi i}{3}})$

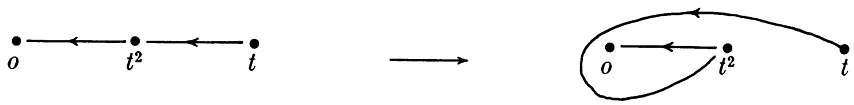
shows that $\begin{cases} \delta_1 \to -\delta_2 \\ \delta_2 \to \delta_1 + \delta_2 \end{cases}$, where

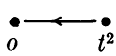
$\delta_1 =$ 

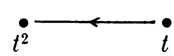
$\delta_2 =$ 

(δ_1, δ_2 generate E_1). Thus $A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $A^6 = I$, as has been found from the differential equation.

(e_3) Locally, $\{V_t\}$ is given by $y^2 = x(x - t)(x - t^2)$. Then the picture



shows that $\begin{cases} \delta_1 \to \delta_1 \\ \delta_2 \to \delta_2 + 2\delta_1 \end{cases}$, where $\delta_1 =$ 

, $\delta_2 =$ 

. In this case, T is not finite order on E_1.

(2) In case the new singularities of V_0 are isolated, we fiber the V_t by hypersurfaces, using the Lefschetz method. Then we may inductively determine T. For example, it is easy to determine that if $\{V_t\}$ is given locally by $x^2 + y^2 + z^2 = t$, and if δ is the 2-cycle Im x = Im y = Im z = $^1/_2$Im t, then $\delta \to -\delta$. This agrees with the differential equation result.

(3) In case the new singular locus $Z \subset V_0$ is nonsingular, then a transverse section of Z in W looks like a pencil acquiring an isolated singularity. Then, from (1) and (2) and the homology theory of fiber bundles, we may frequently find T. For example, if dim $V = 2$ and $Z \subset V_0$ is a nonsingular double curve of genus p, then H. Clemens, in his Berkeley thesis, has determined that

$$A = 2p\left\{\left(\begin{array}{cccc|c} \multicolumn{4}{c|}{\overbrace{\hphantom{1\quad 1\quad 1\quad 1}}^{2p}} & \\ 1 & 1 & & & \\ & 1 & 0 & & \\ & & 1 & 1 & \\ & & & 1 & 0 \\ \hline & & & 0 & I \end{array}\right)\right. \quad \text{or } A = 2p - 2\left\{\left(\begin{array}{cccc|c} \multicolumn{4}{c|}{\overbrace{\hphantom{1\quad 1\quad 1\quad 1}}^{2p-2}} & \\ 1 & 1 & & & \\ & 1 & 0 & & \\ & & 1 & 1 & \\ & & & 1 & 0 \\ \hline & & & & I \end{array}\right)\right.$$

depending on whether the inverse image $\tilde{Z}$ of Z in the normalization $\tilde{V}$ of V_0 is disconnected or not.

(4) In general, Z is the disjoint union of (open) nonsingular subvarieties, $Z = Z_1 \cup \ldots \cup Z_e$, and one may try to apply (3) to each Z_j and then hope to fit things together. For example if dim $V_0 = 2$ and locally $Z = Z_1 \cup Z_2 \cup Z_3$ (triple point),

then $$T^3 = \left(\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \\ \hline & 0 & & * \end{array}\right)$$ so T_{E_q} is not of finite order.

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[1] Griffiths, P. A., these PROCEEDINGS, in press.